Diversity of Core-Periphery Structure in Real Networks

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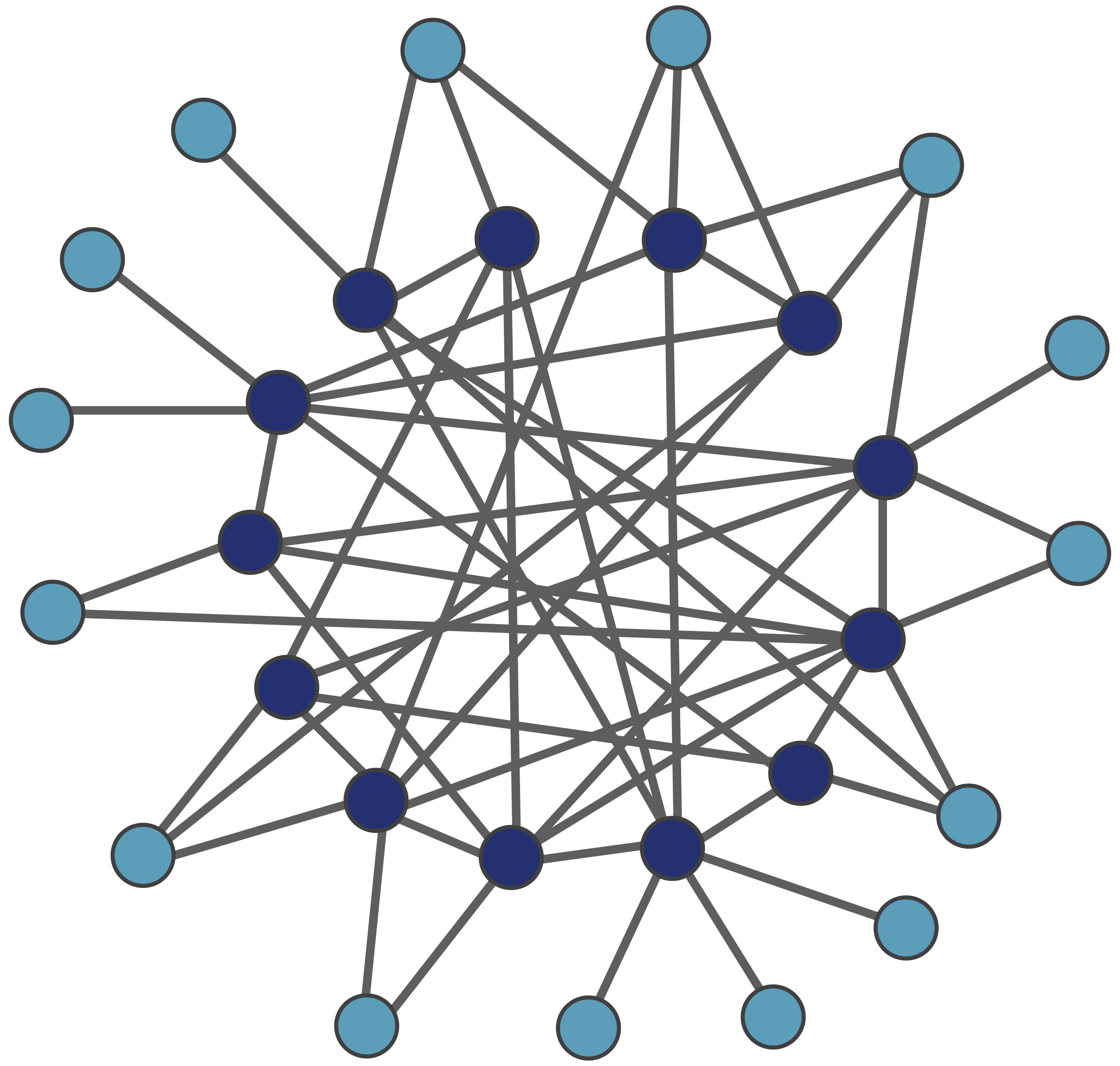
What is core-periphery structure?

Core-Periphery Structure

Two-Block Model

"Core nodes are adjacent to other core nodes, core nodes are adjacent to some periphery nodes, and periphery nodes do not connect with other periphery nodes."

- Borgatti, S.P. & Everett, M.G., 2000



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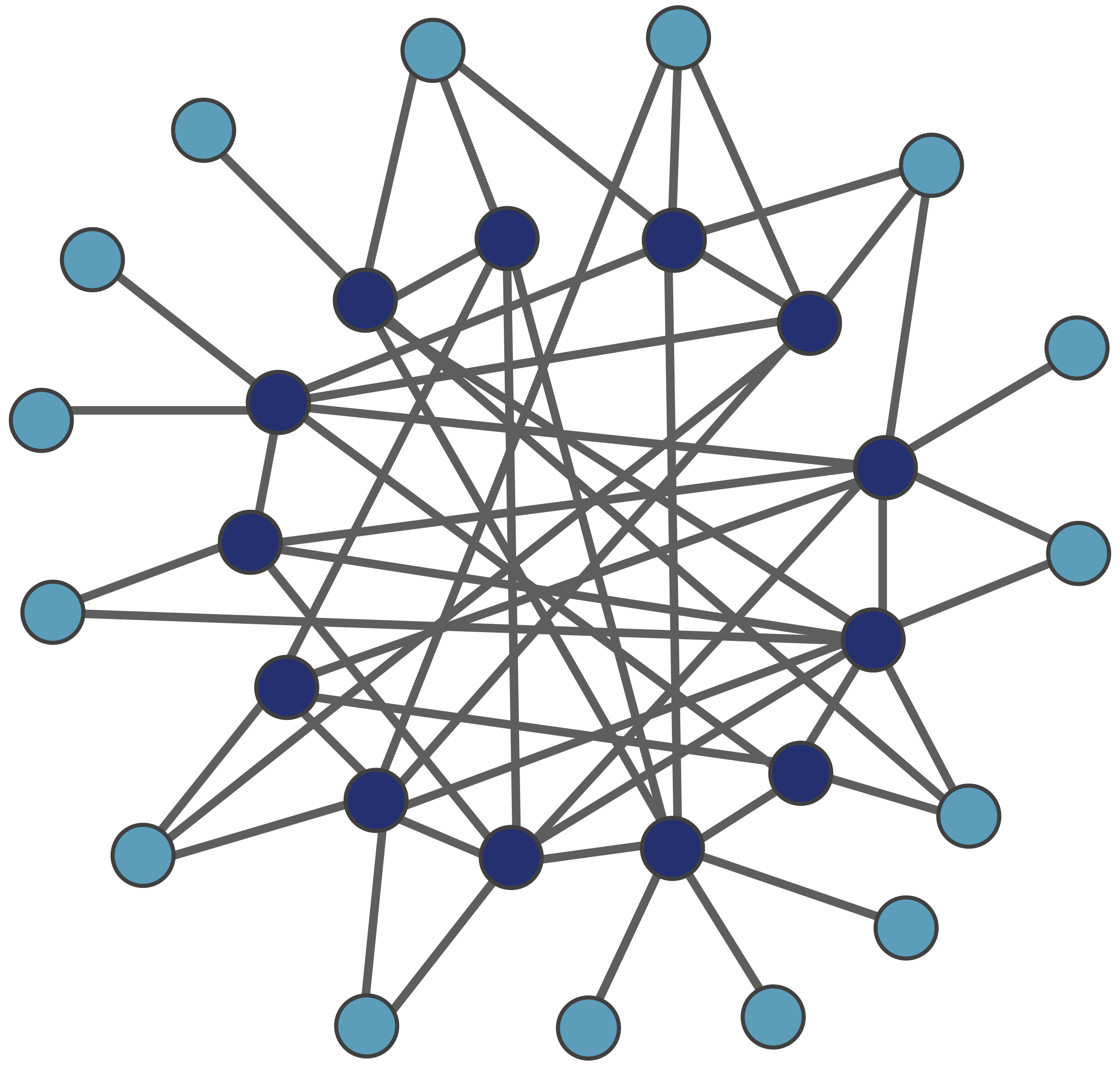

Hub and spokes

Radial

Star-like

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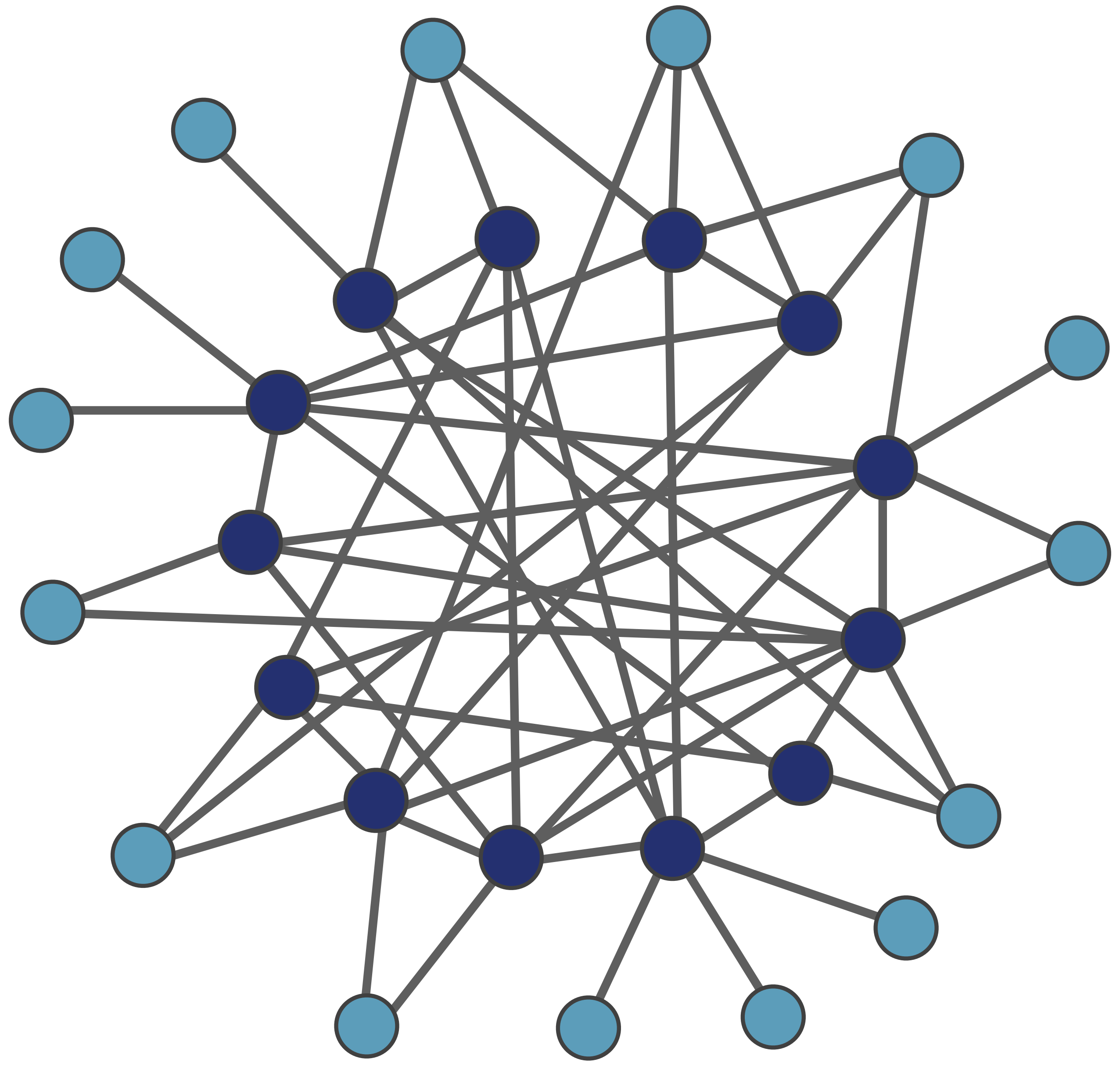

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k-Cores Decomposition

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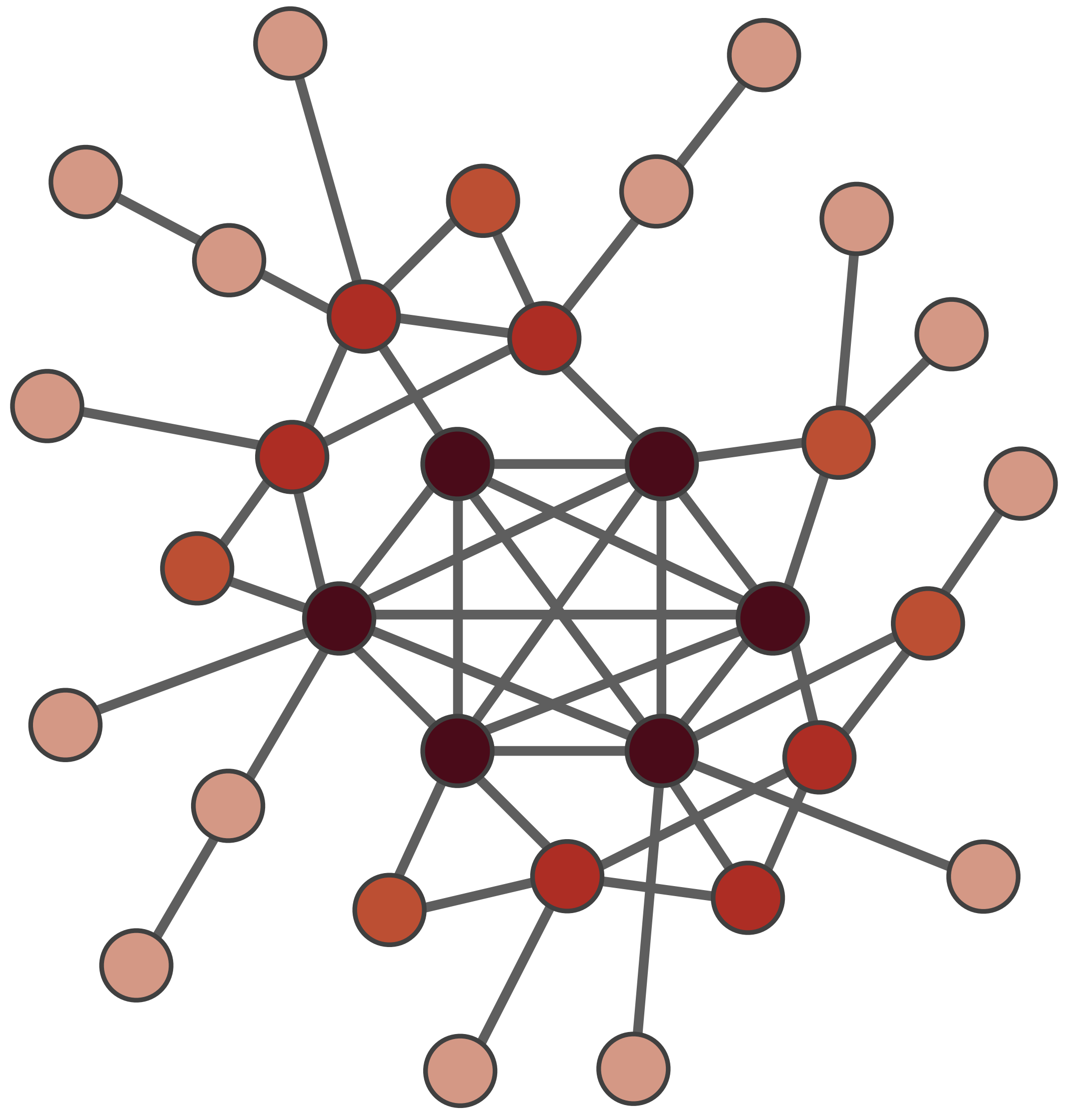

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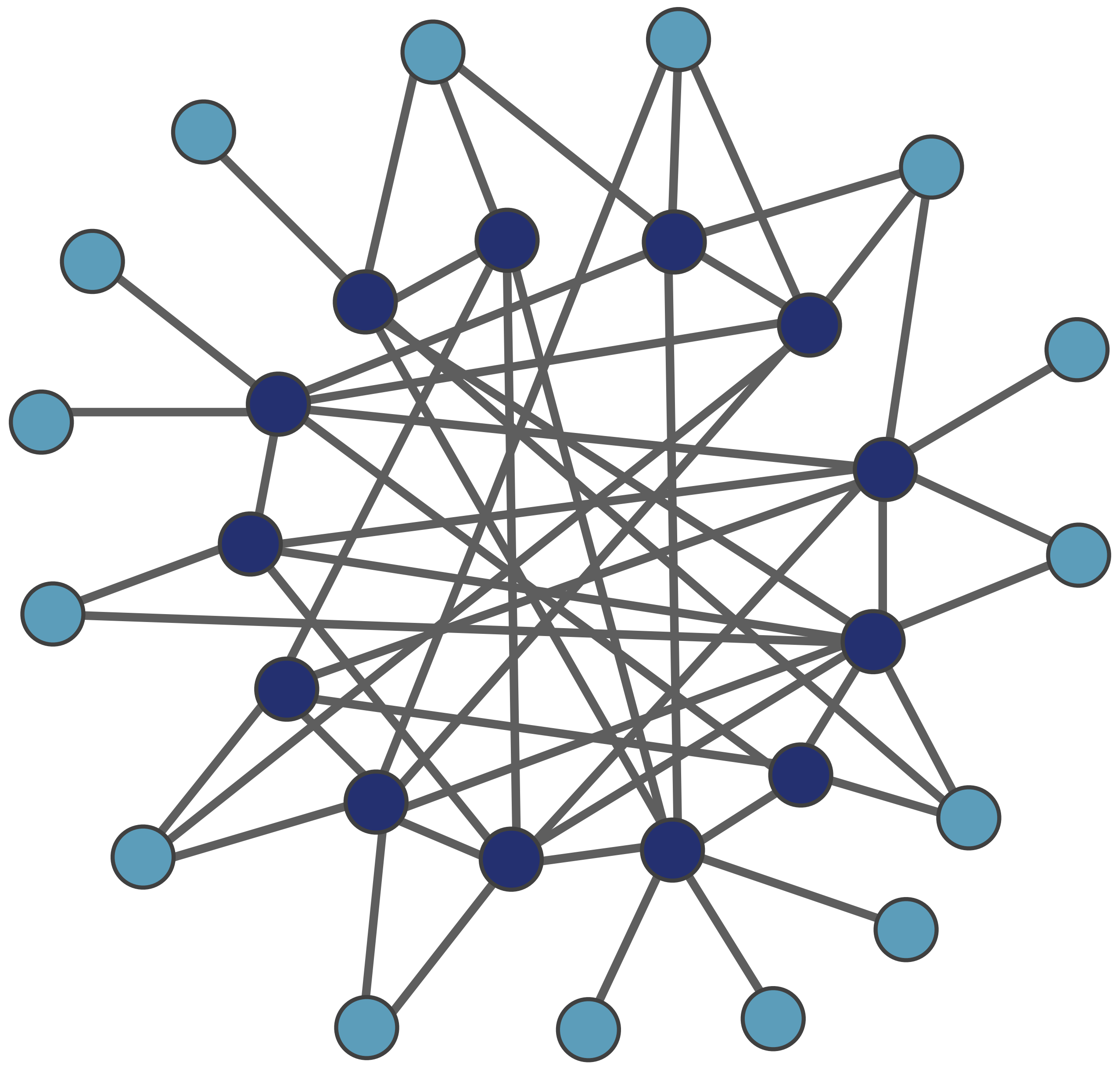

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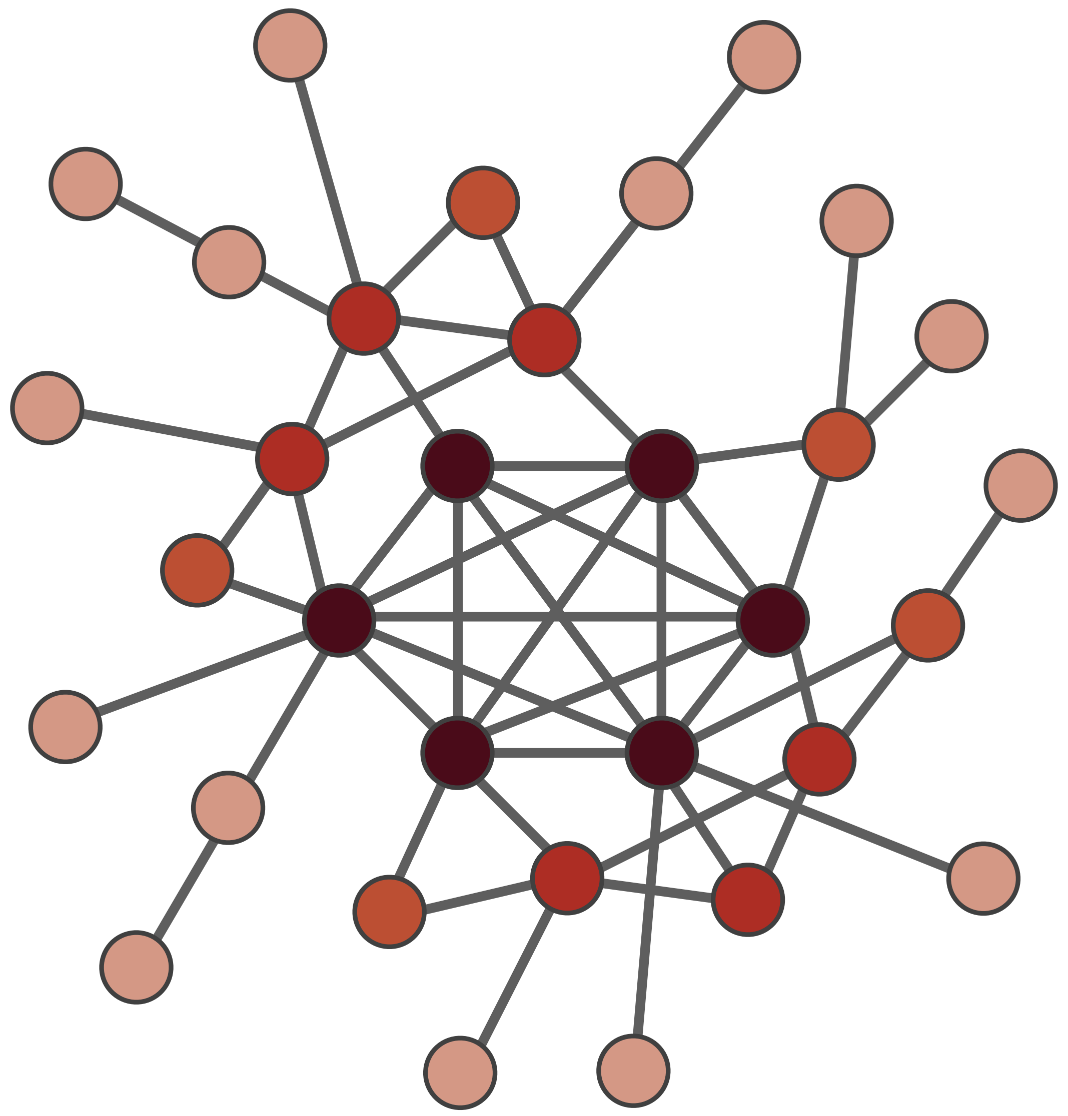


Layers

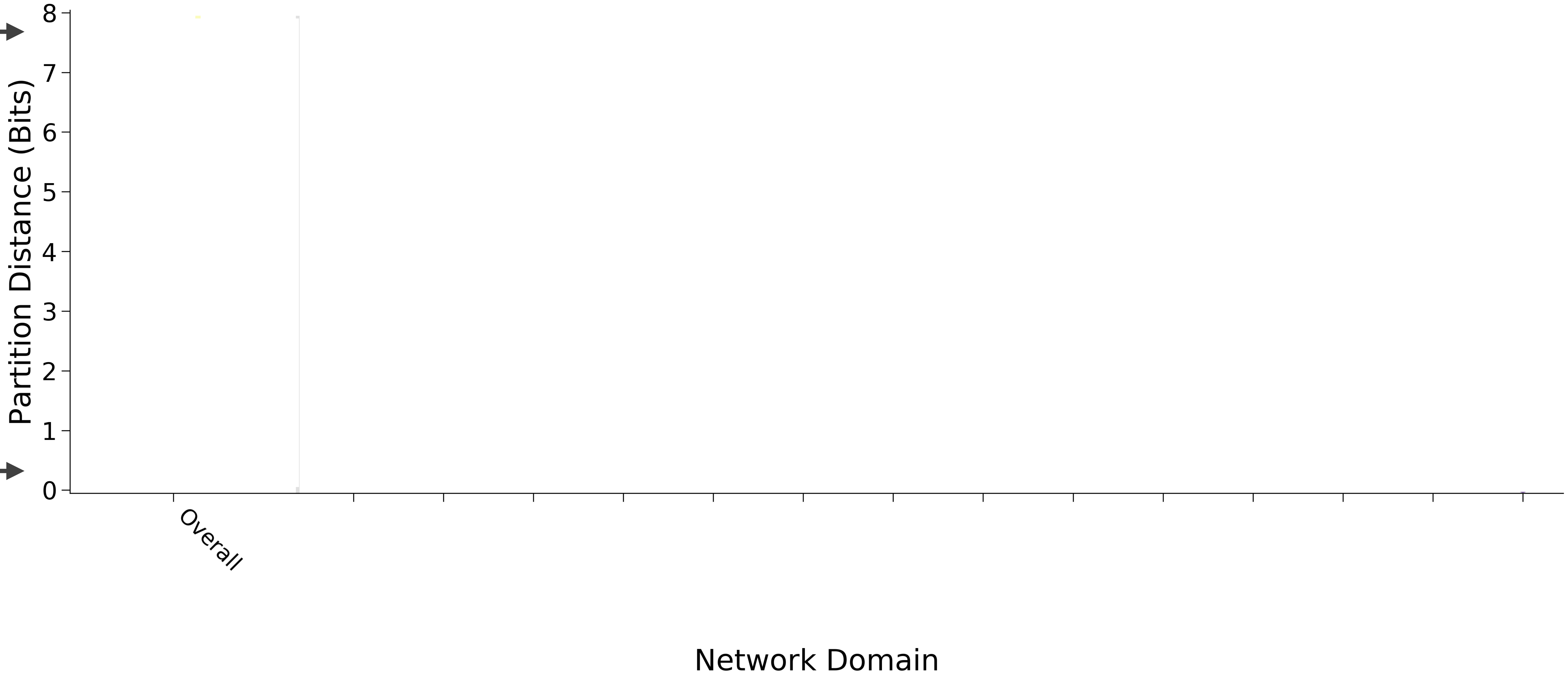
Hierarchy

Shells

Core-Periphery Partition Comparison

Two-block model and *k*-cores partitions are in perfect agreement

Partitions are completely dissimilar



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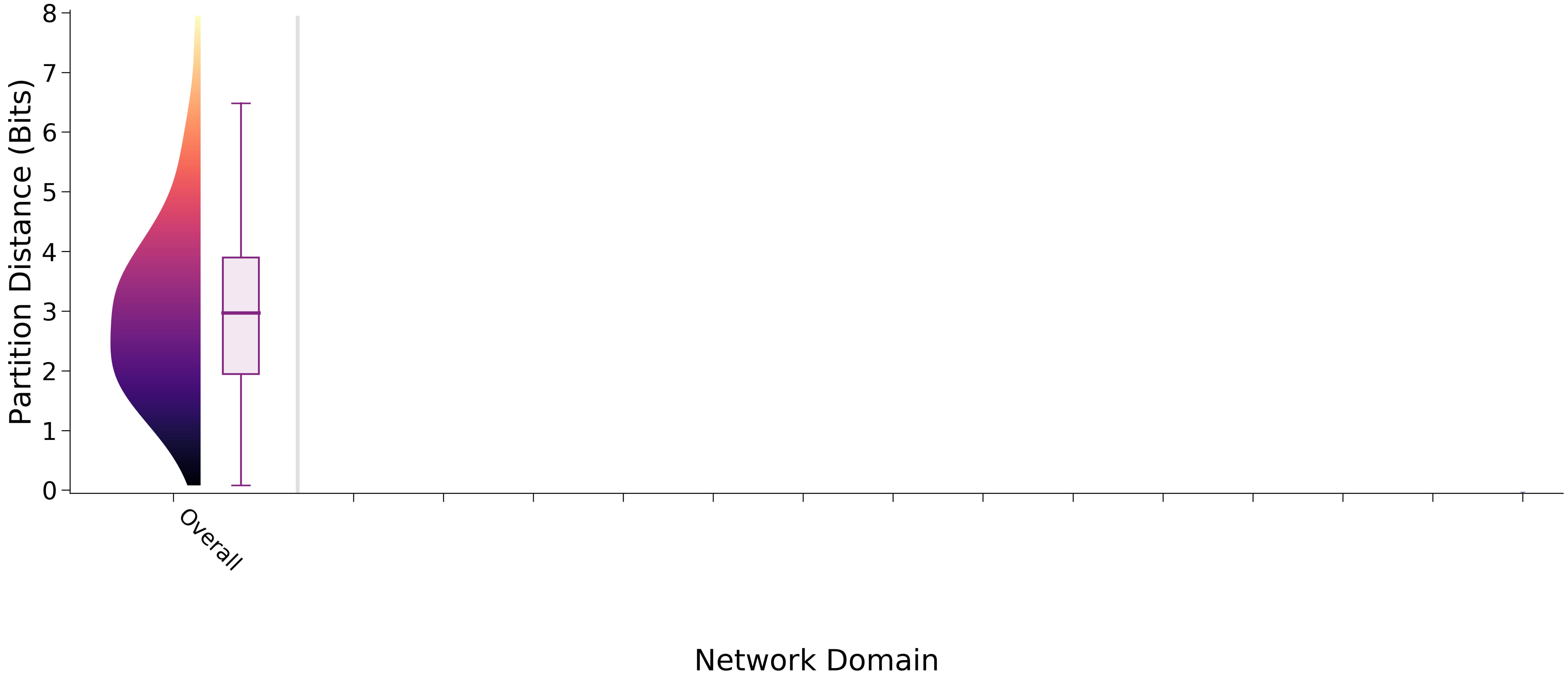

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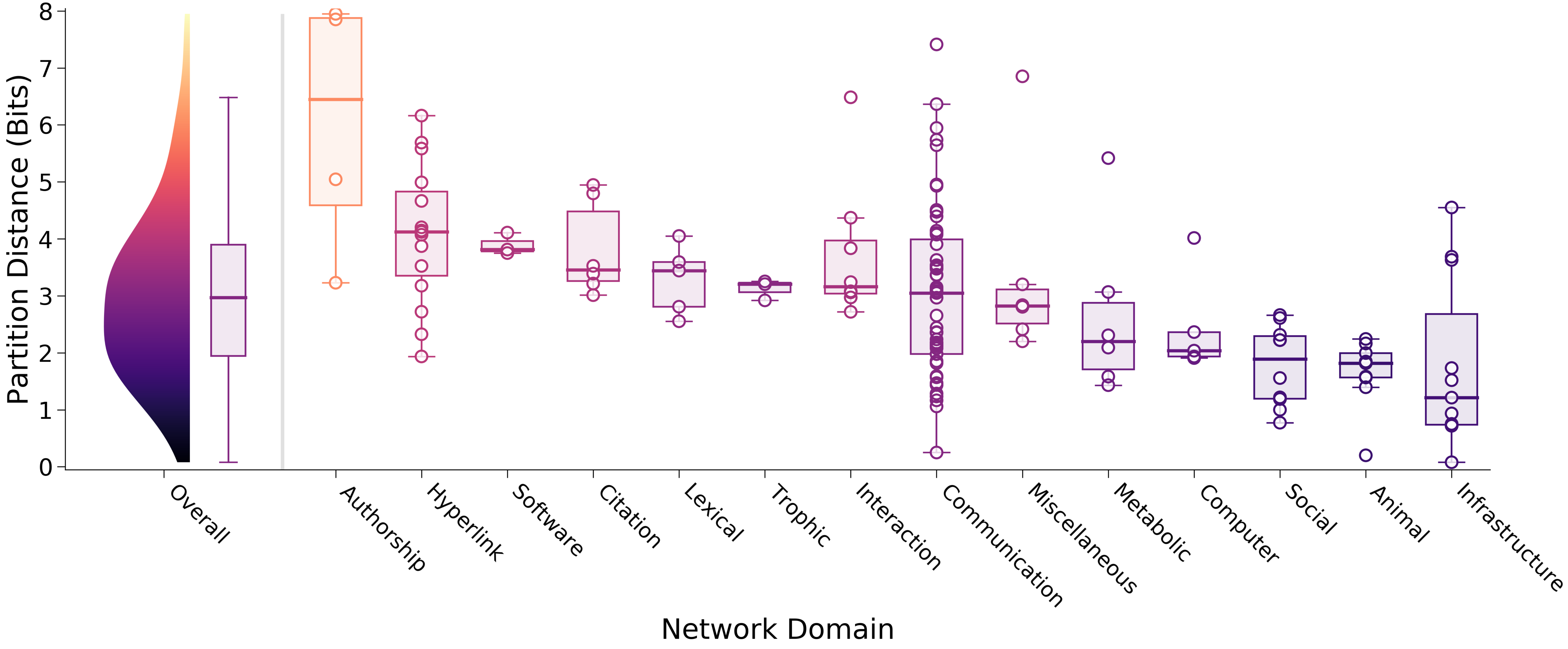

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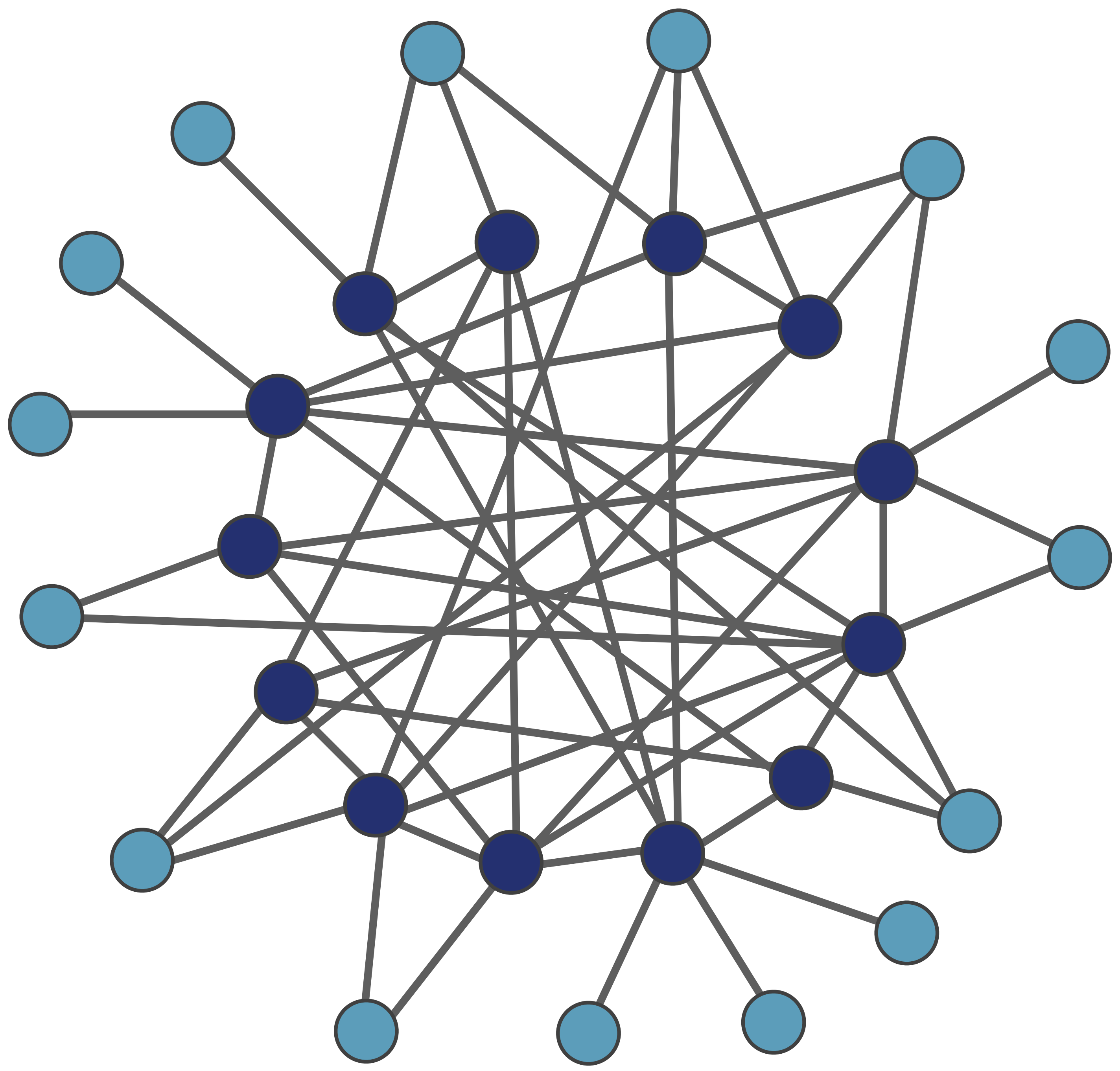
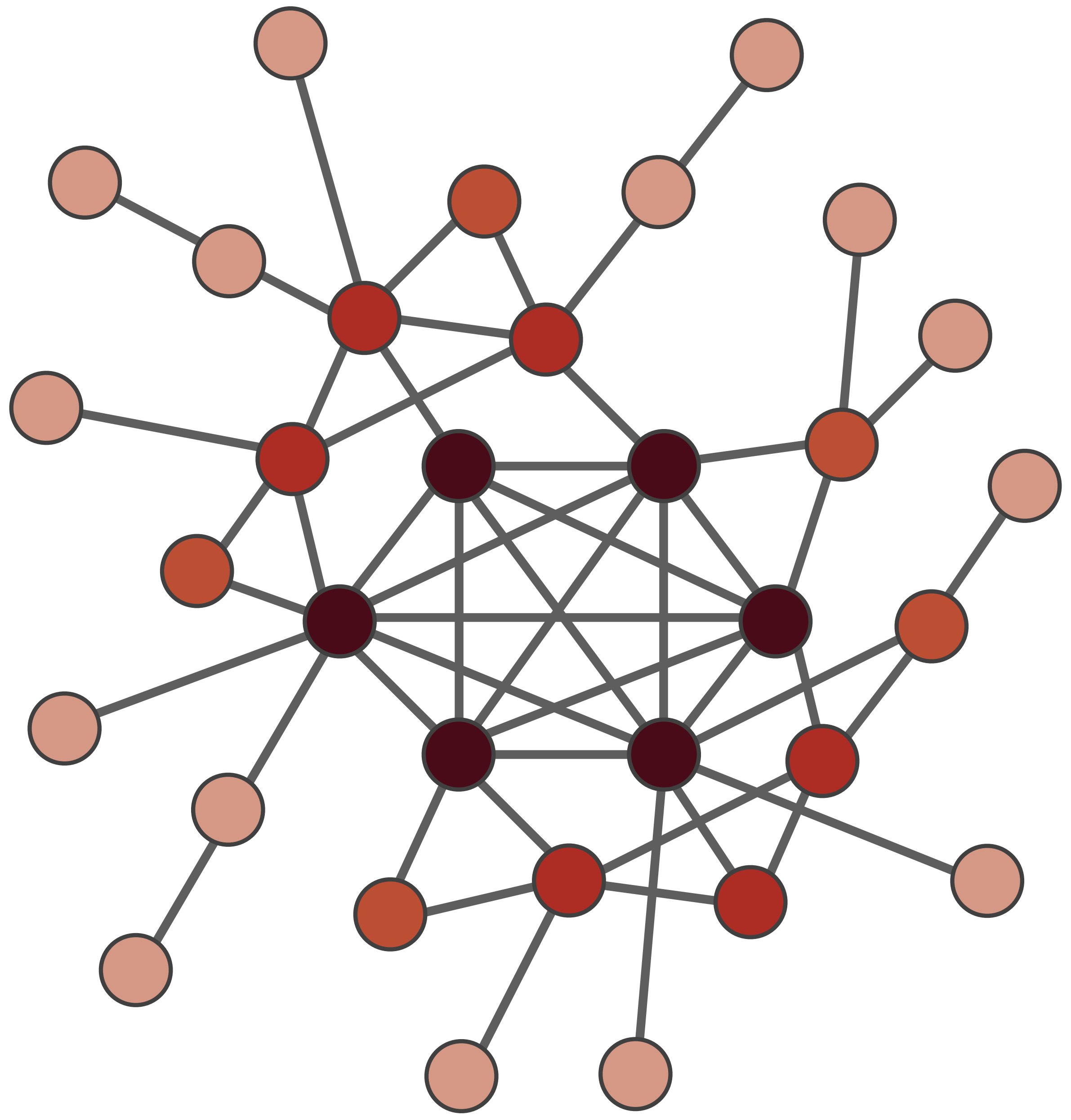
Core-Periphery Typology

The two-block model and the *k*-cores decomposition exemplify a ***typology of core-periphery structure***

Hub-and-spoke



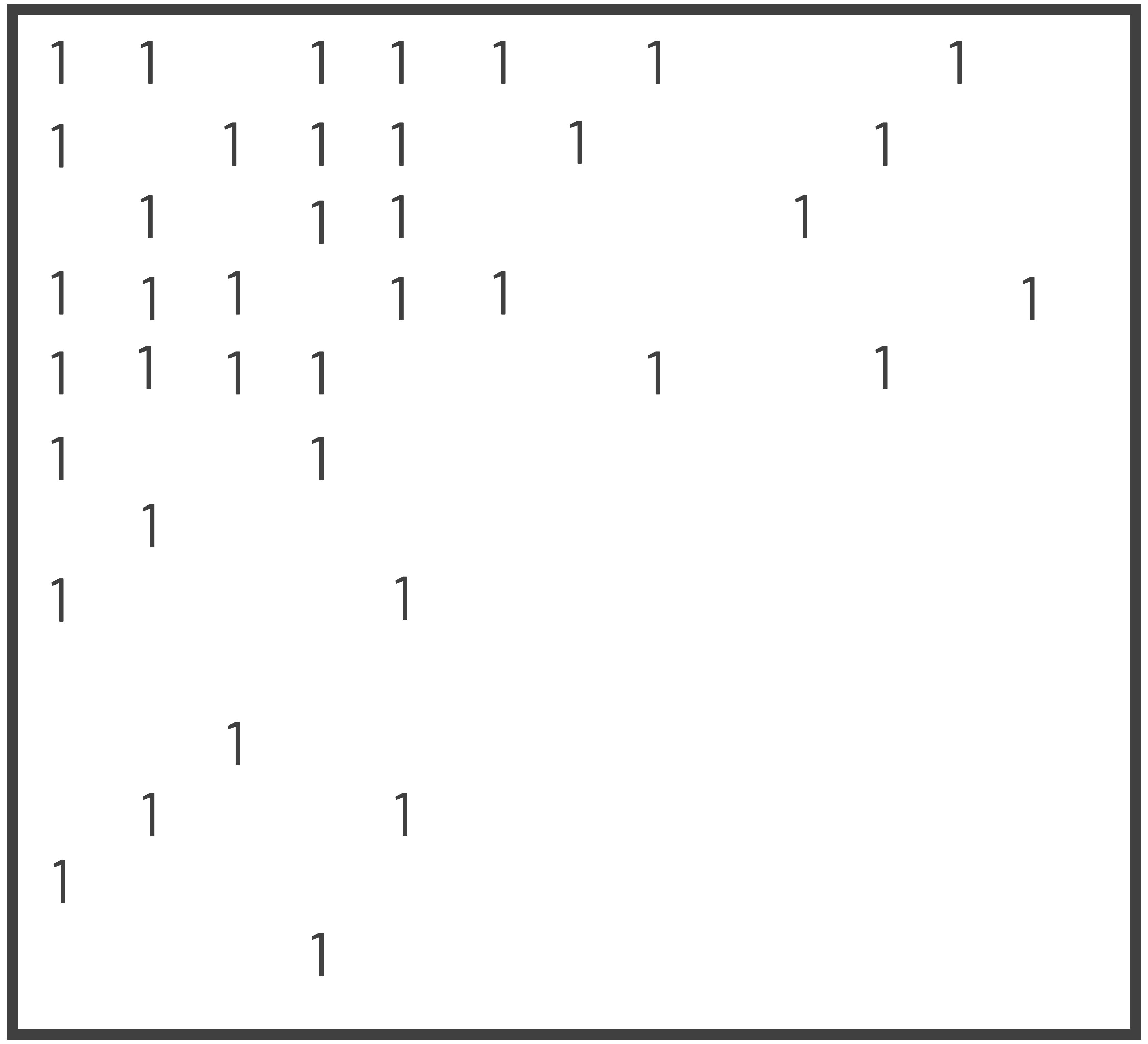
Layered



How do we determine which type of core-periphery structure best describes a network?

Core-Periphery Stochastic Block Models

We can encode our prior notions of core-periphery structure through *Bayesian stochastic block models*



Adjacency matrix

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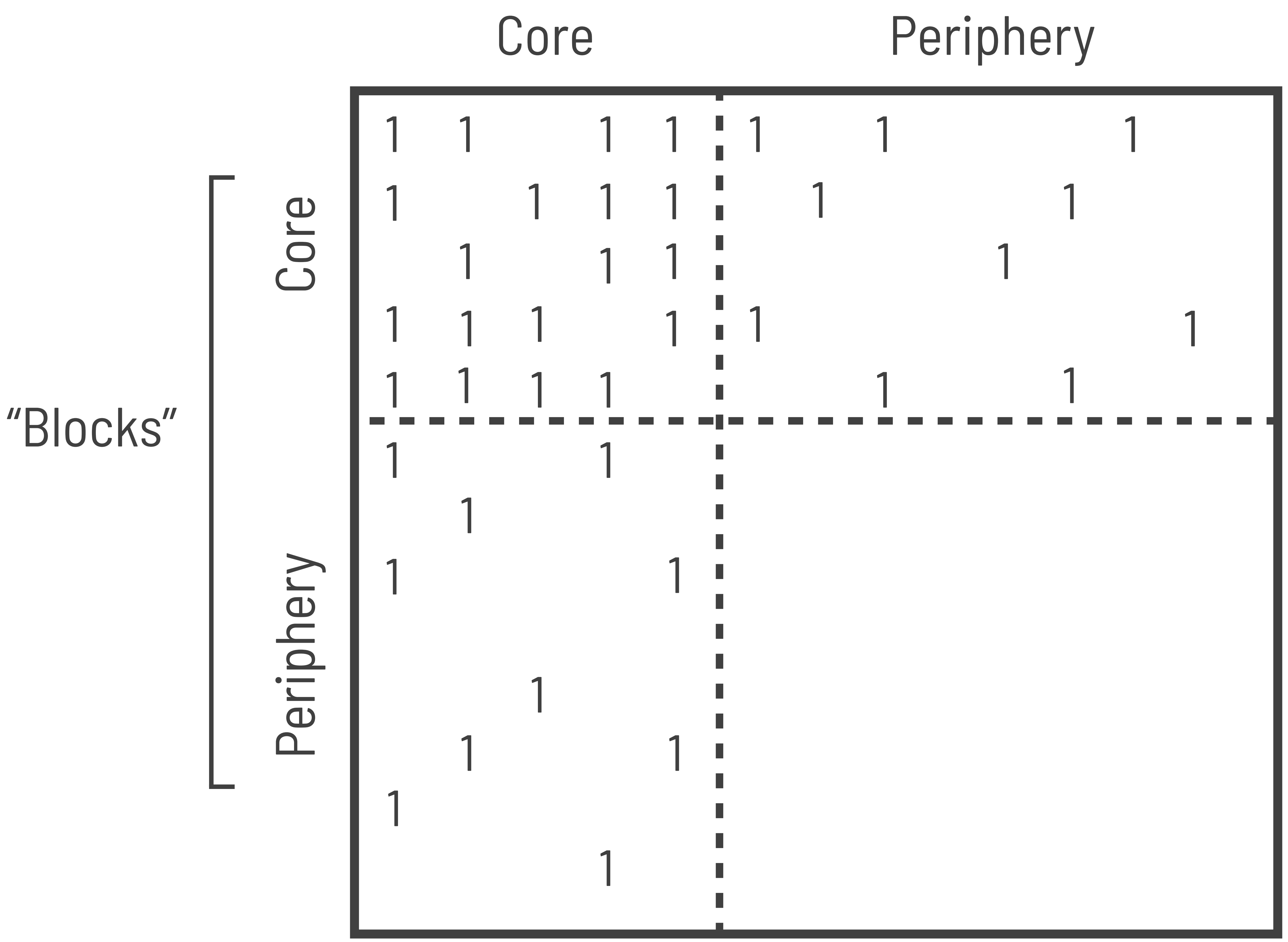

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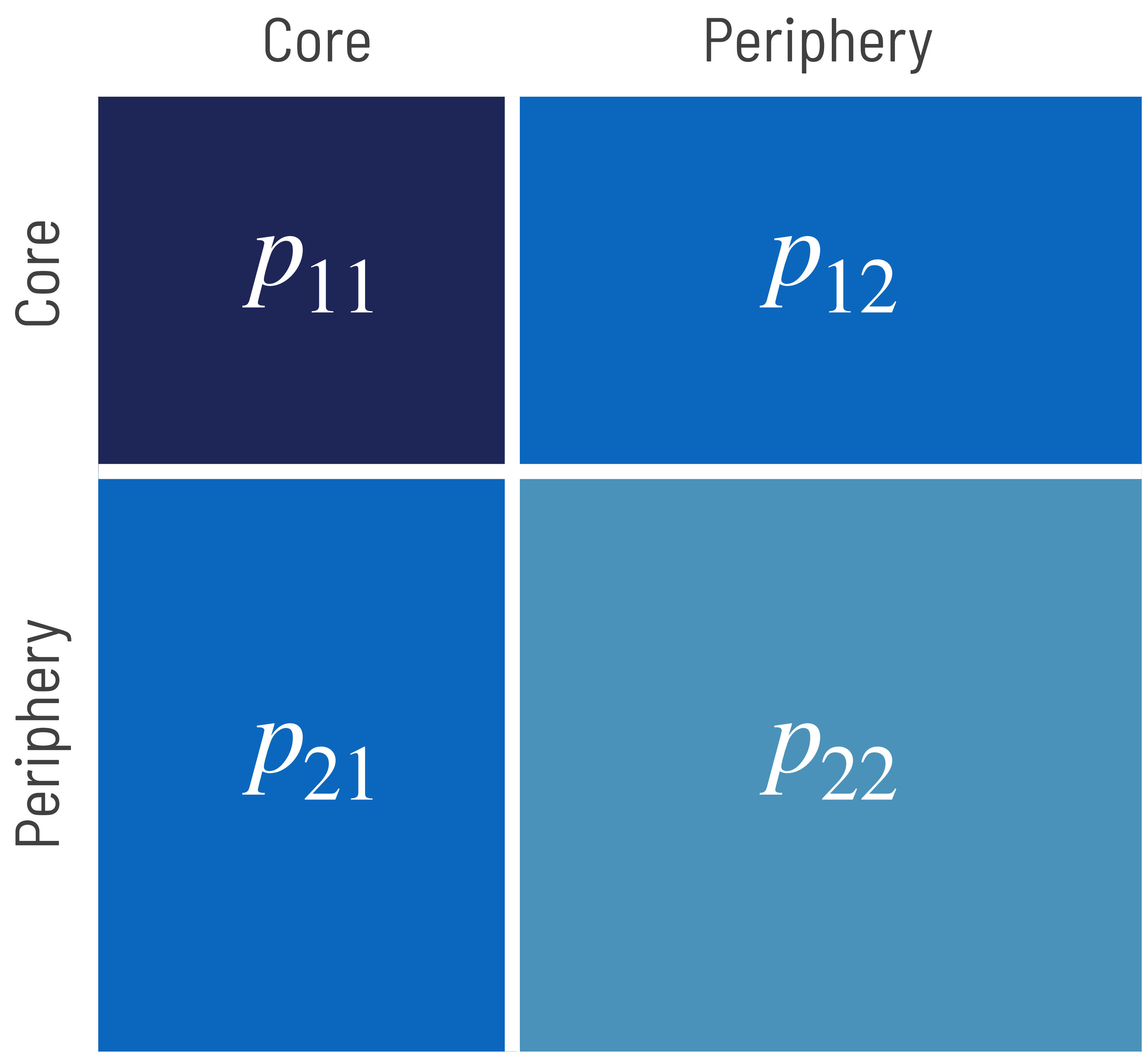

Block matrix

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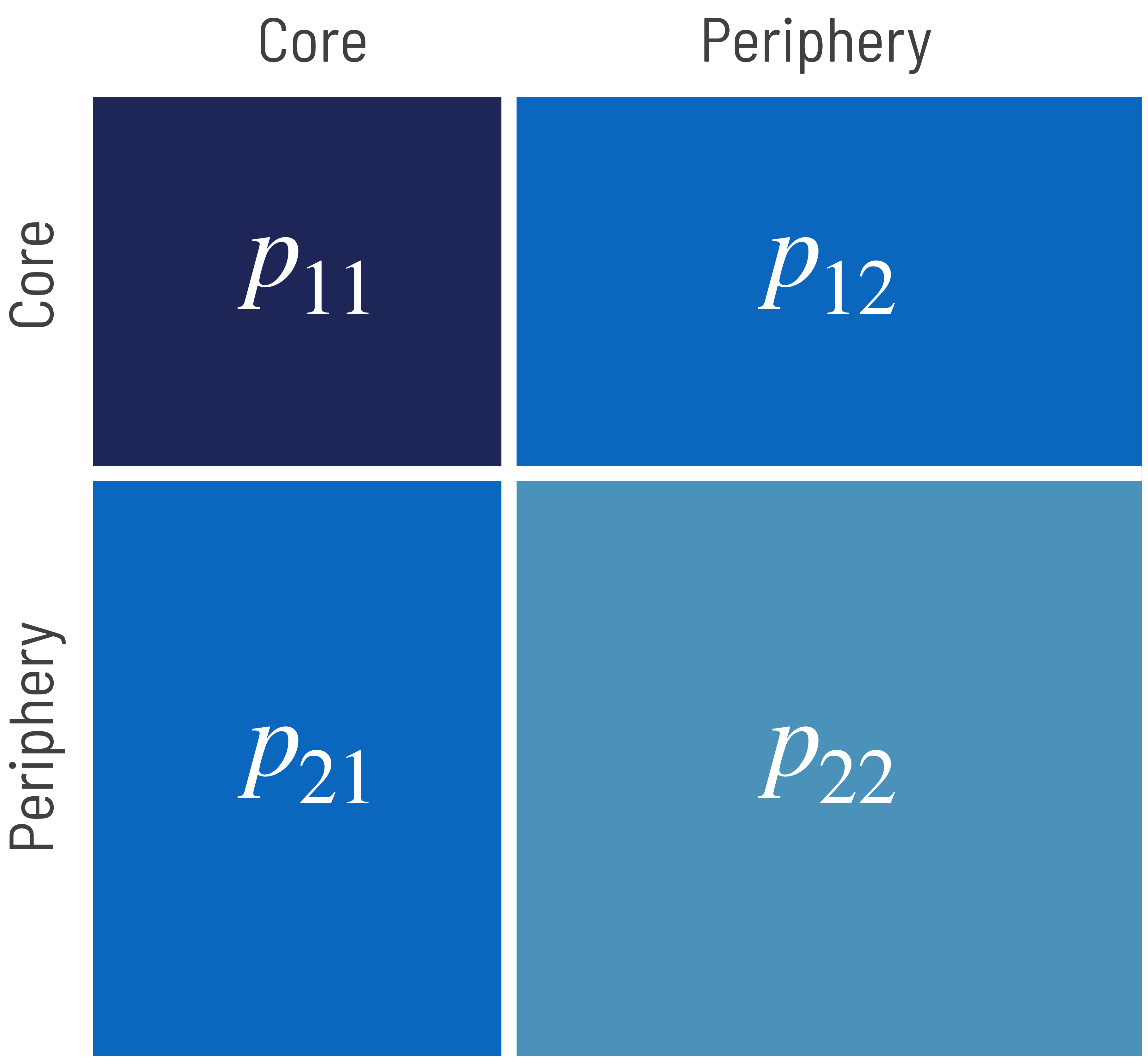
$$P(\theta, \mathbf{p} \mid \mathbf{A})$$



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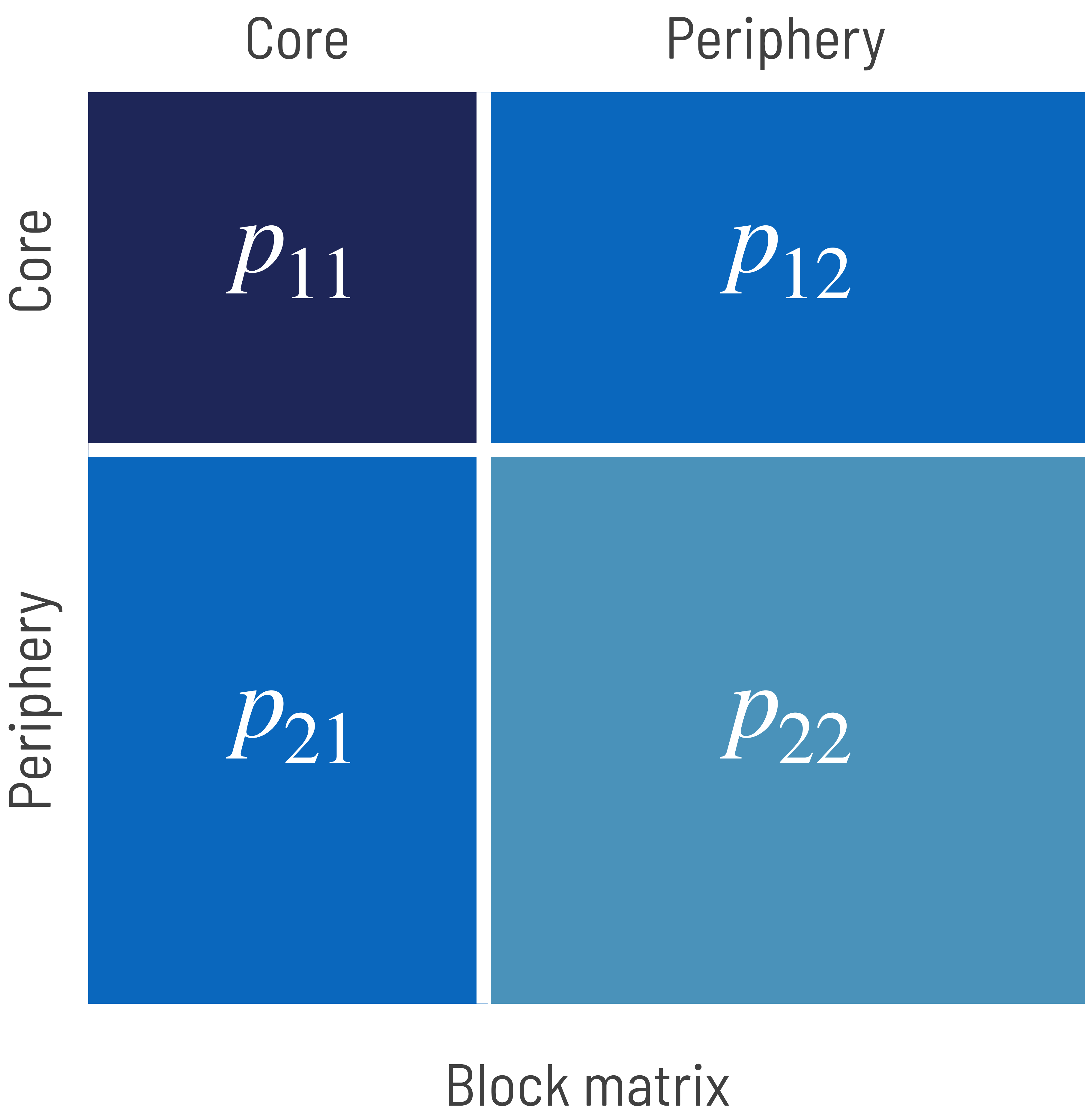
Assignments of nodes to blocks



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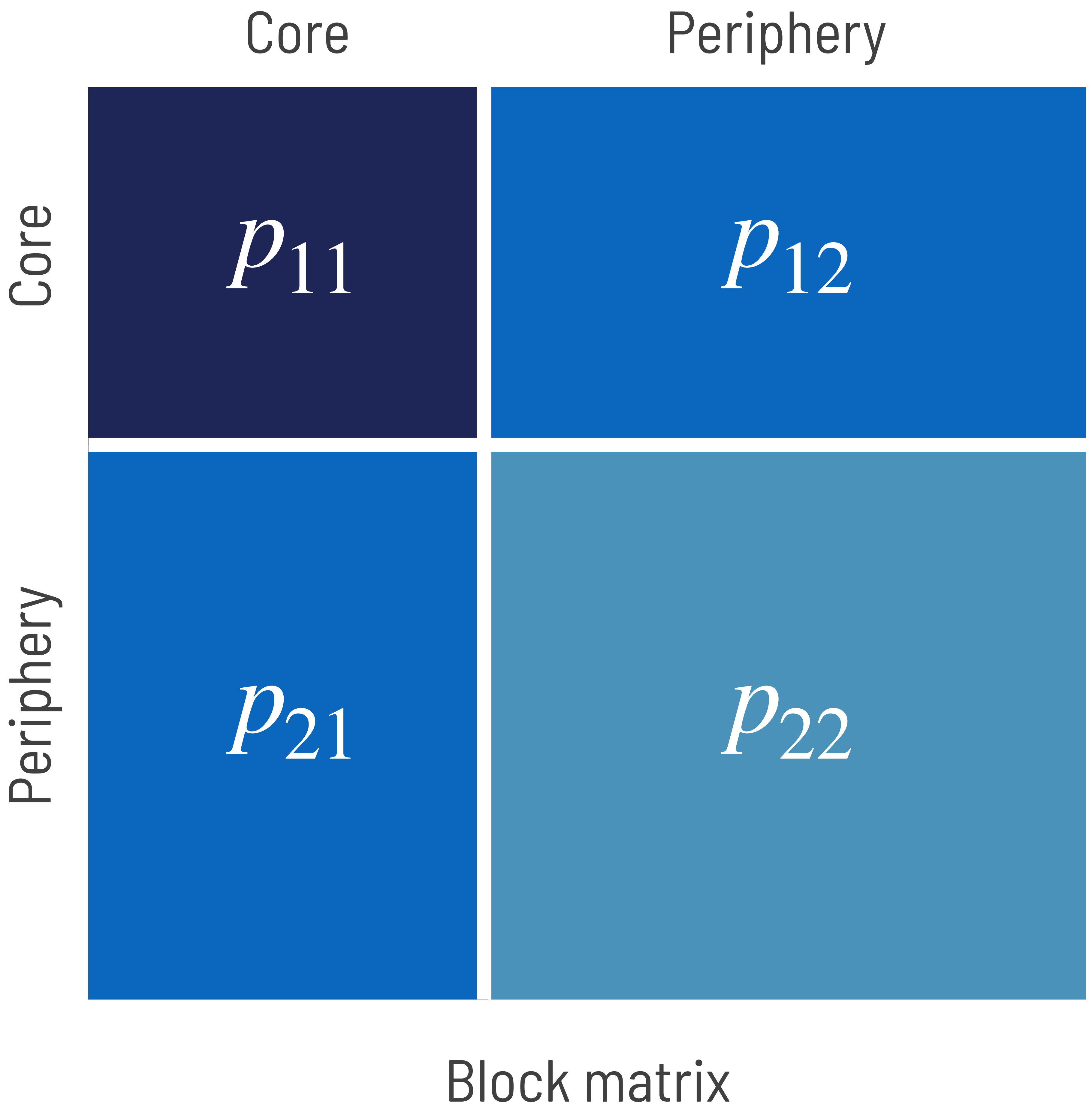

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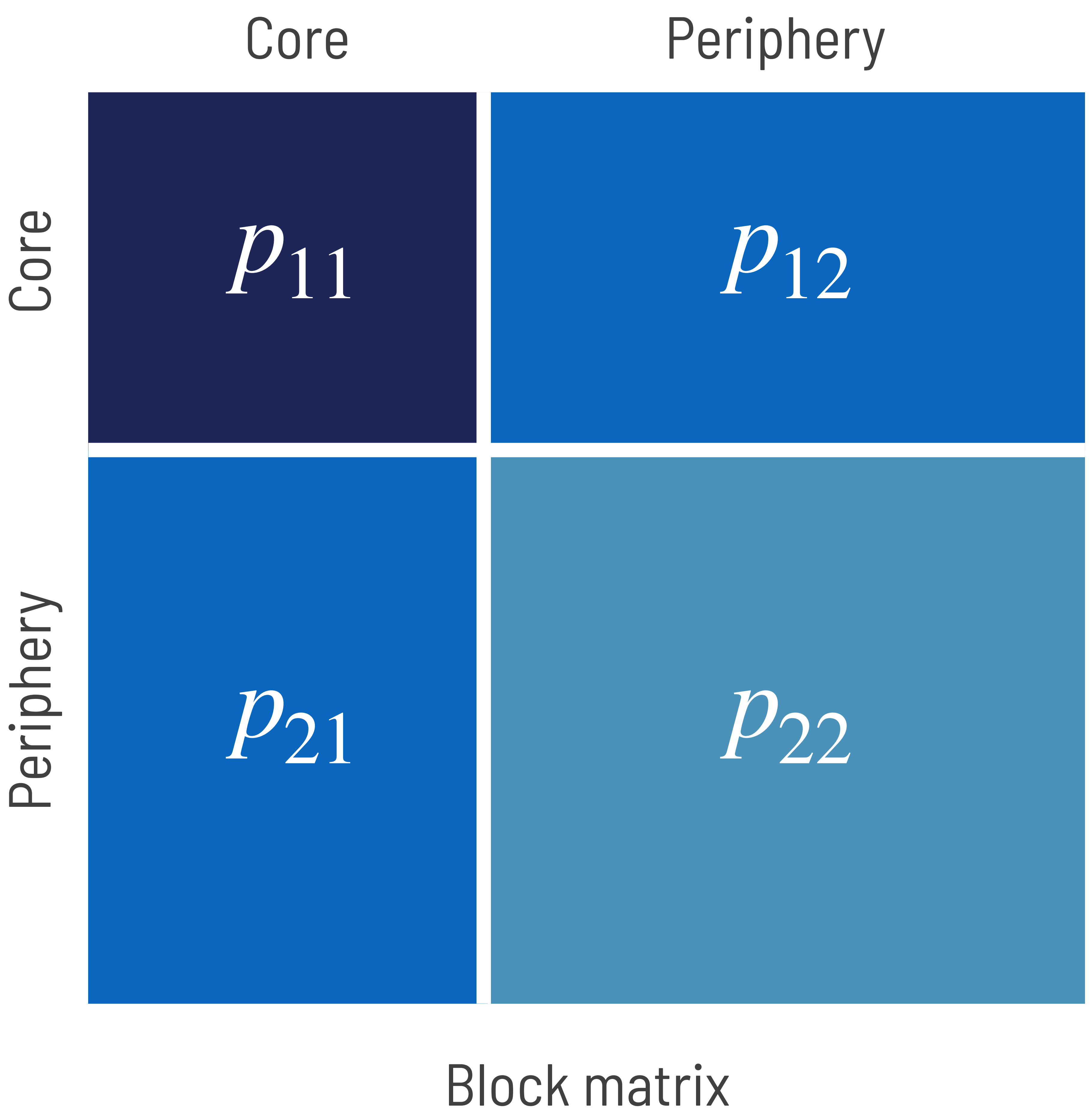
Network data



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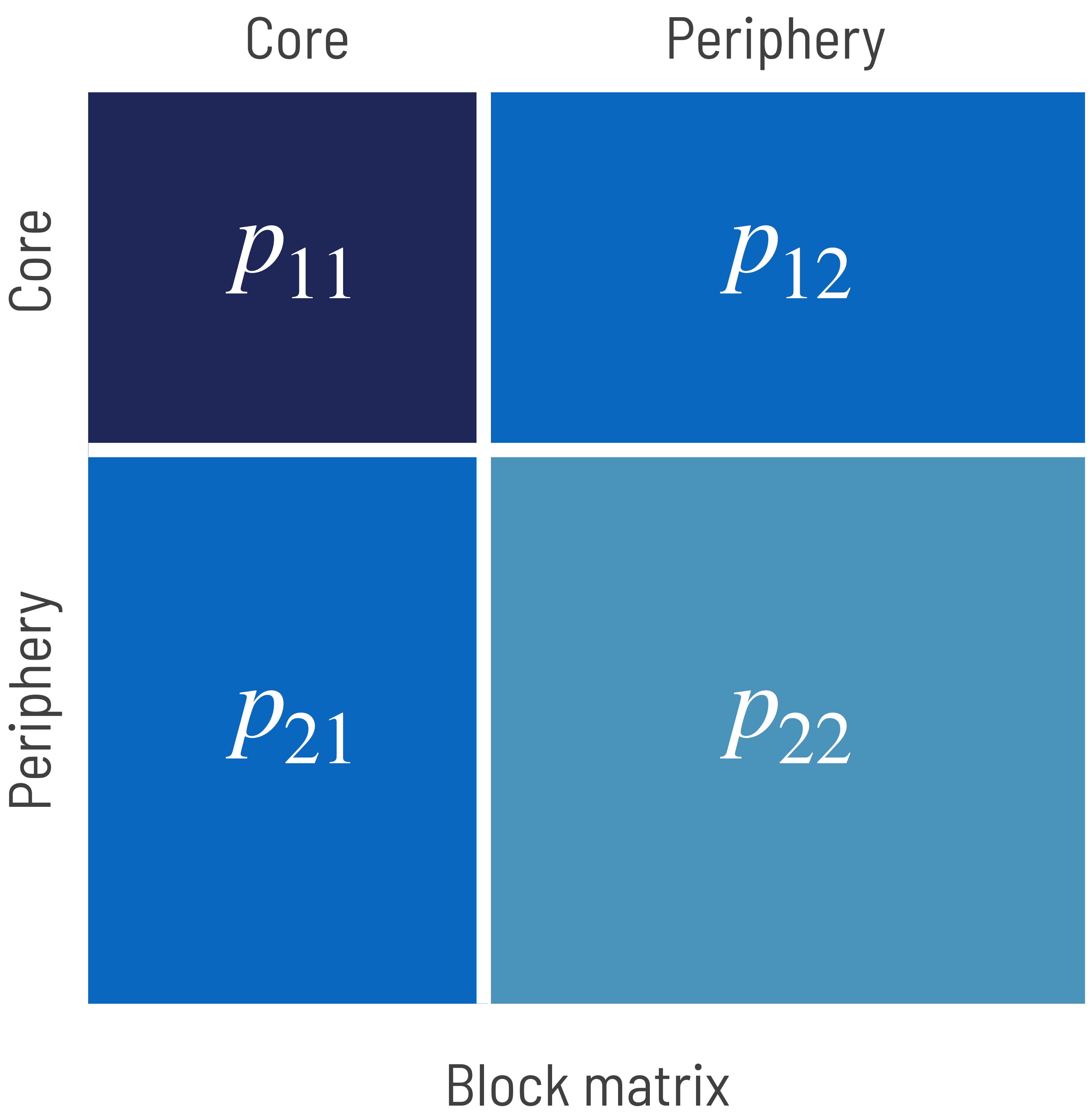
Posterior distribution



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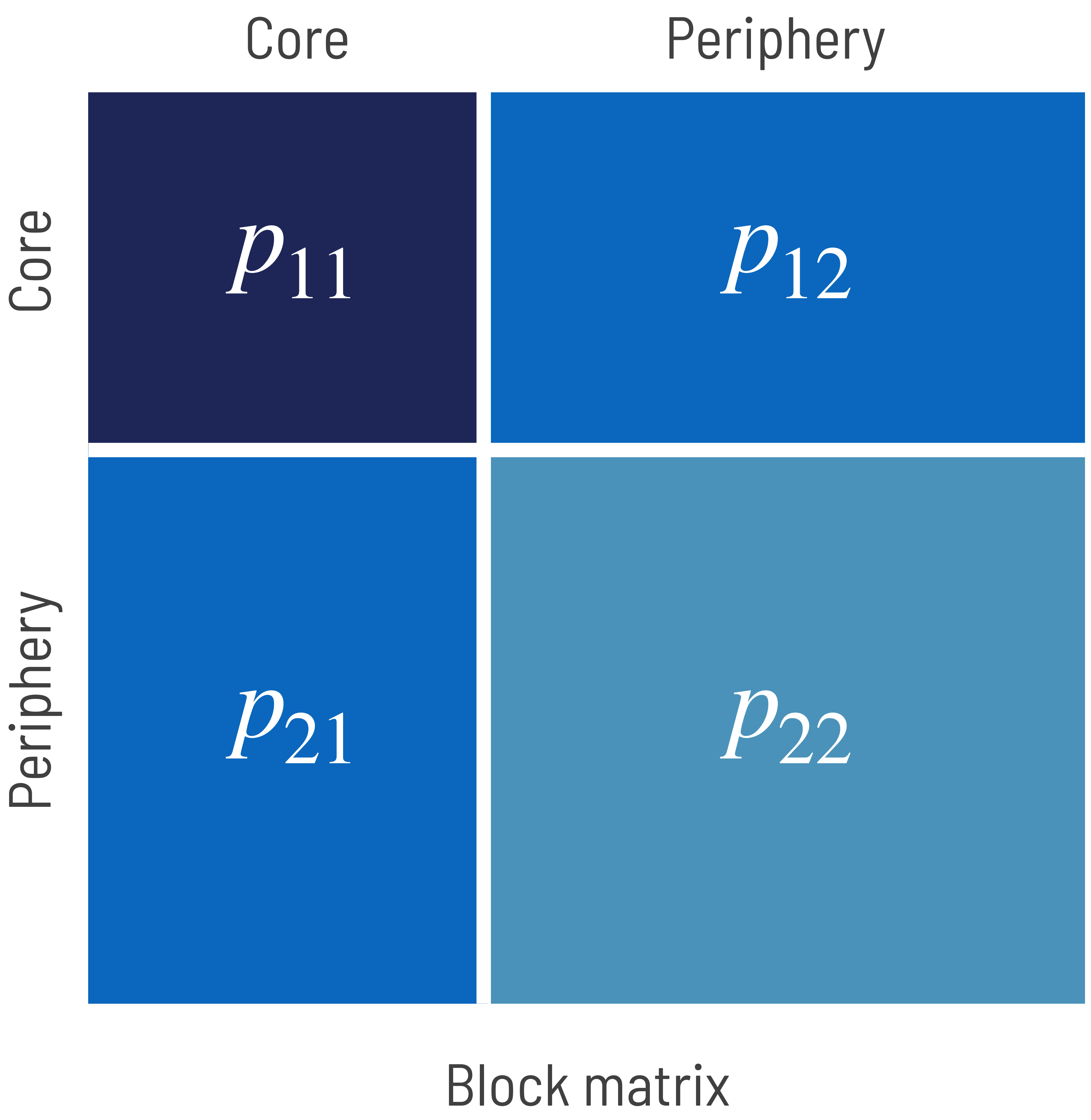

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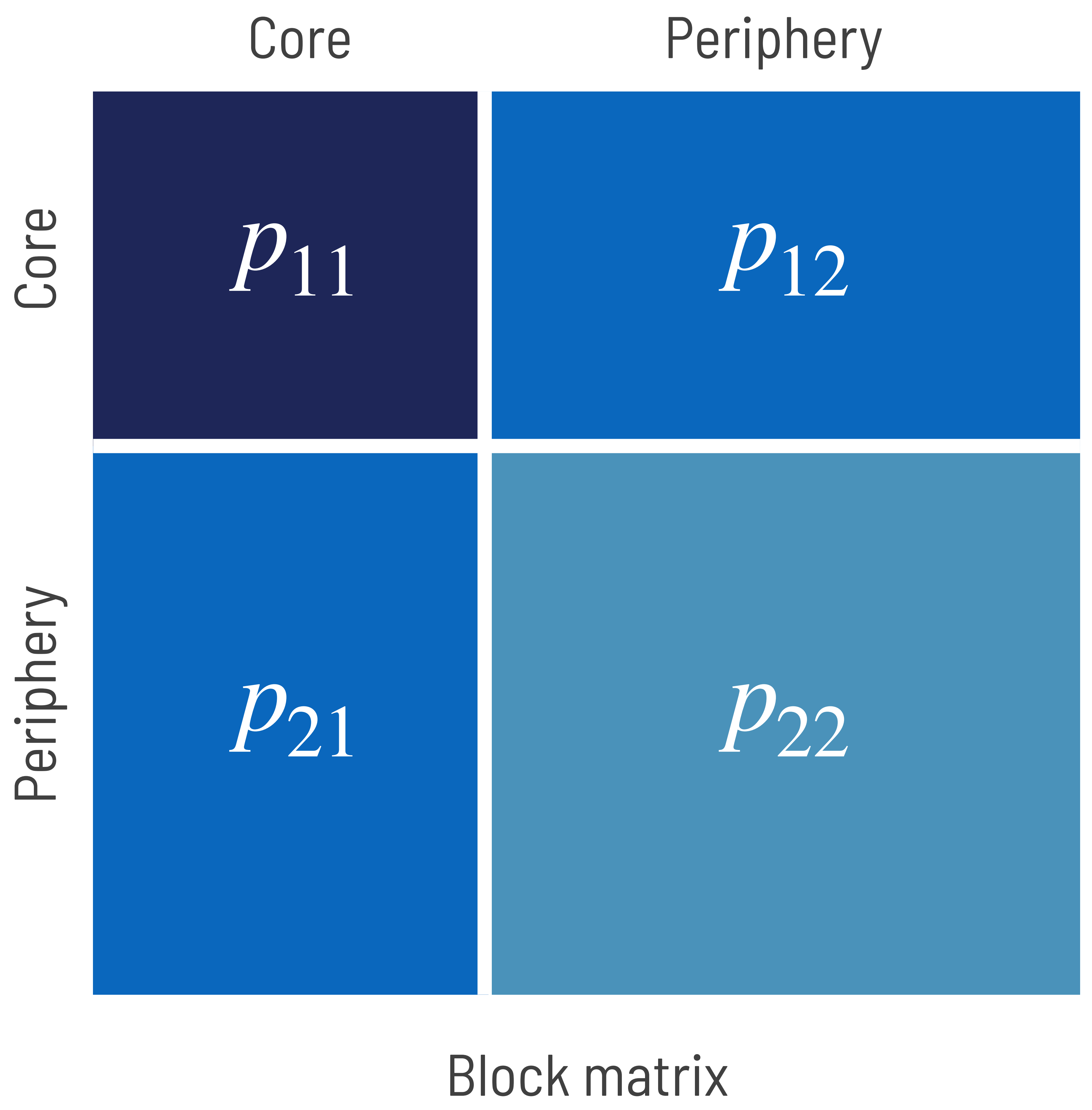
Covered by
prior work



Karrer, B., & Newman, M. E. (2011). Stochastic blockmodels and community structure in networks. *Physical Review E*, *83*(1), 016107.

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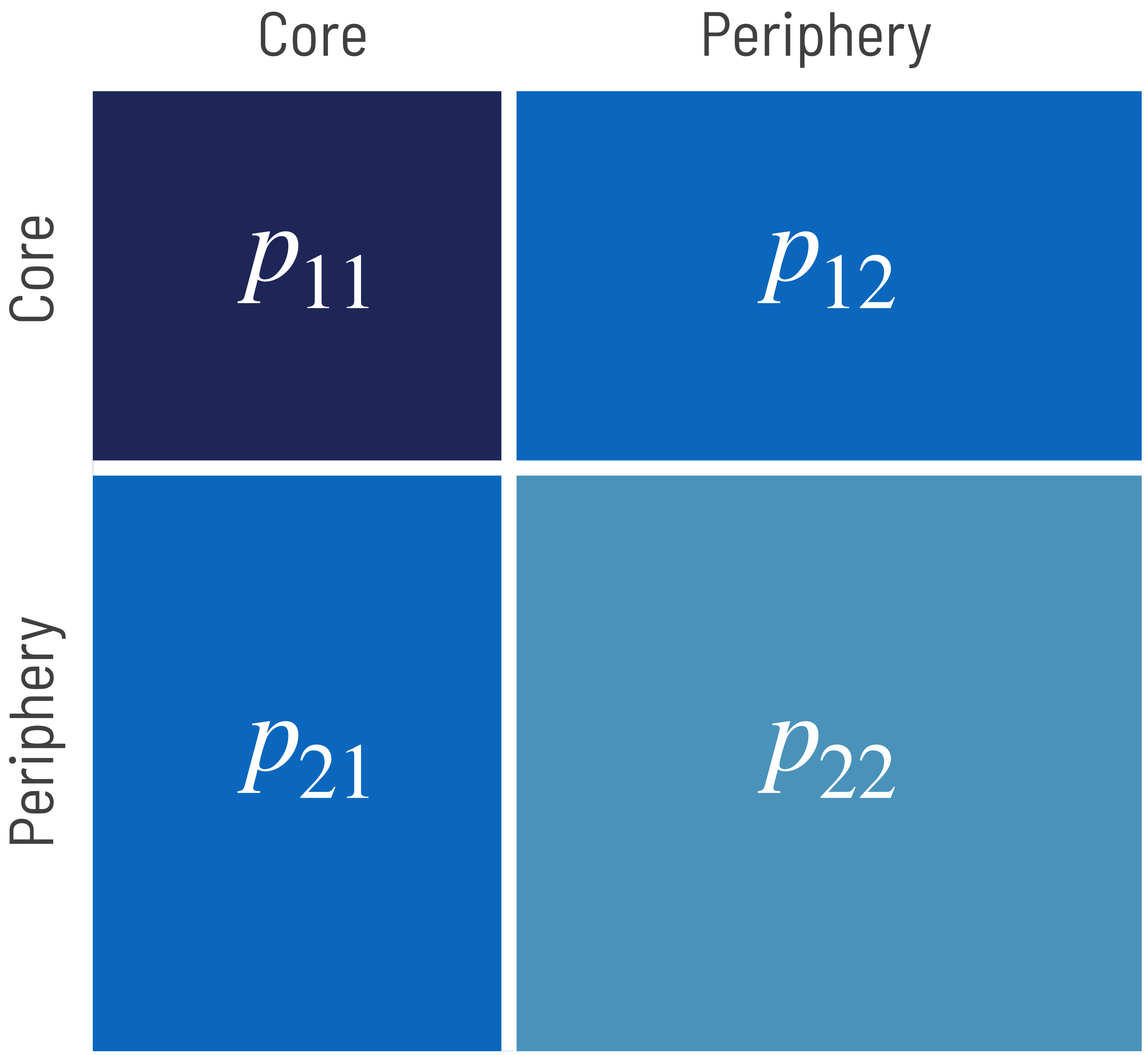
Prior on block matrix



Block matrix

Block Connectivity Priors

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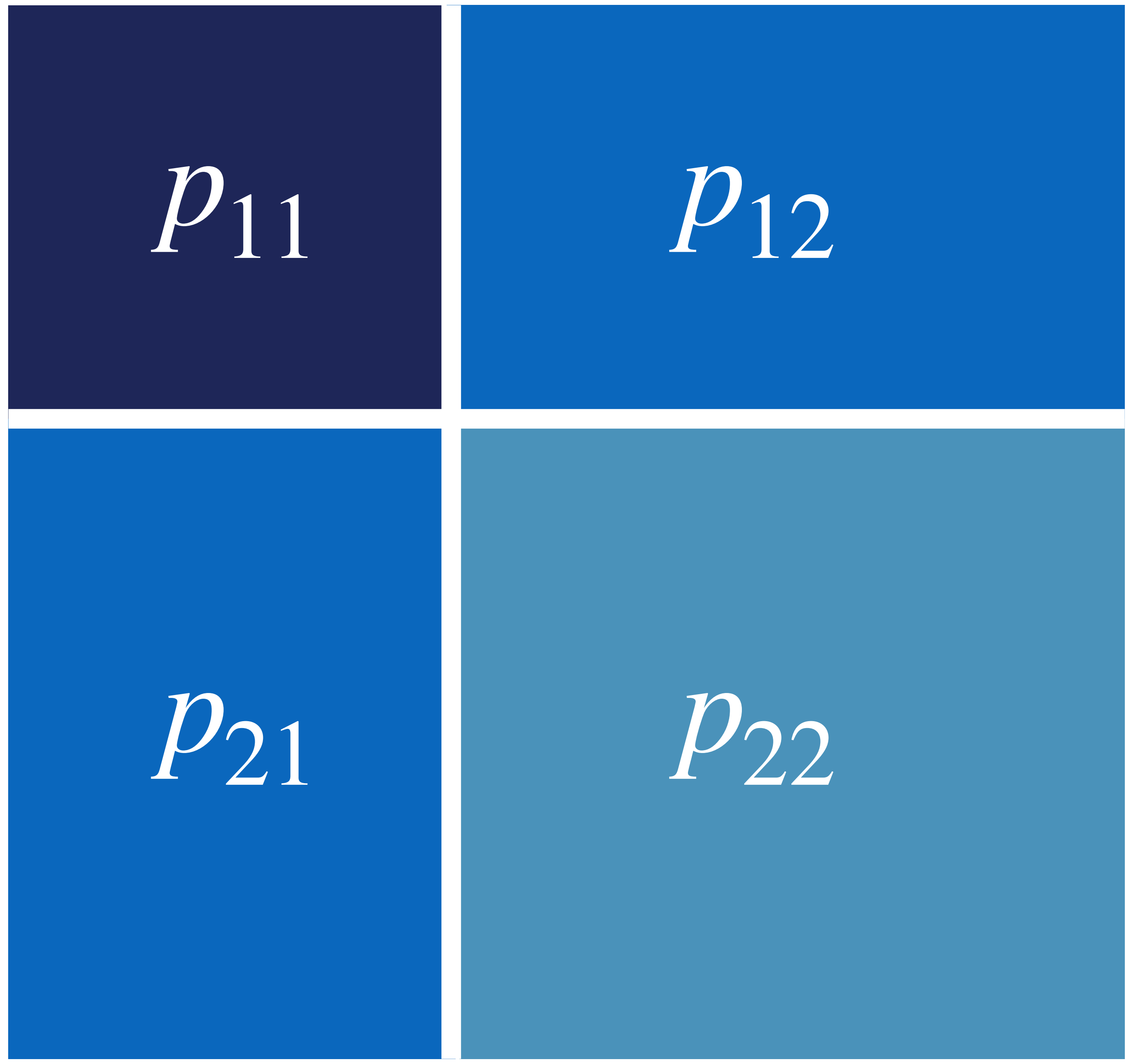
Hub-and-spoke



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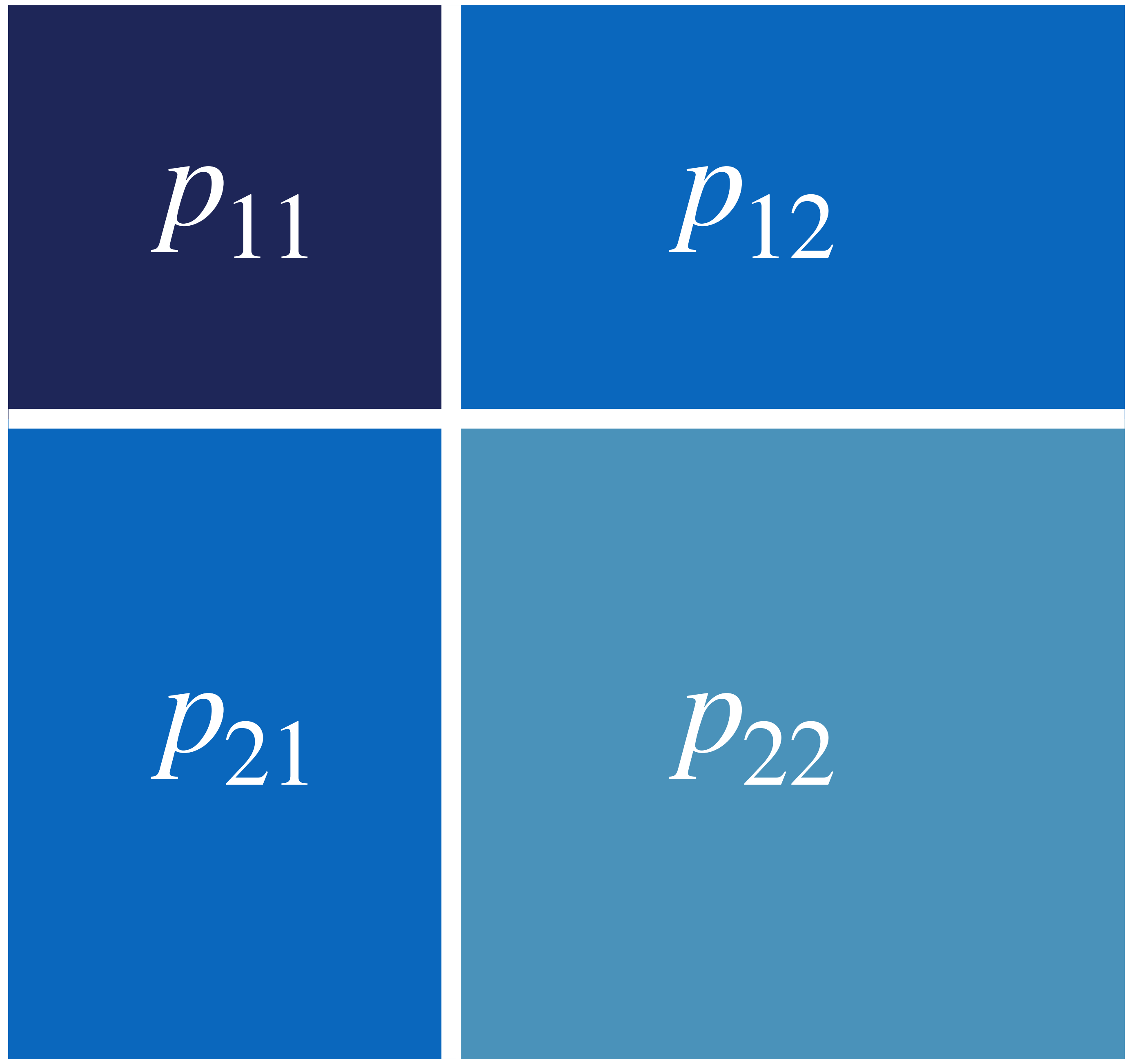

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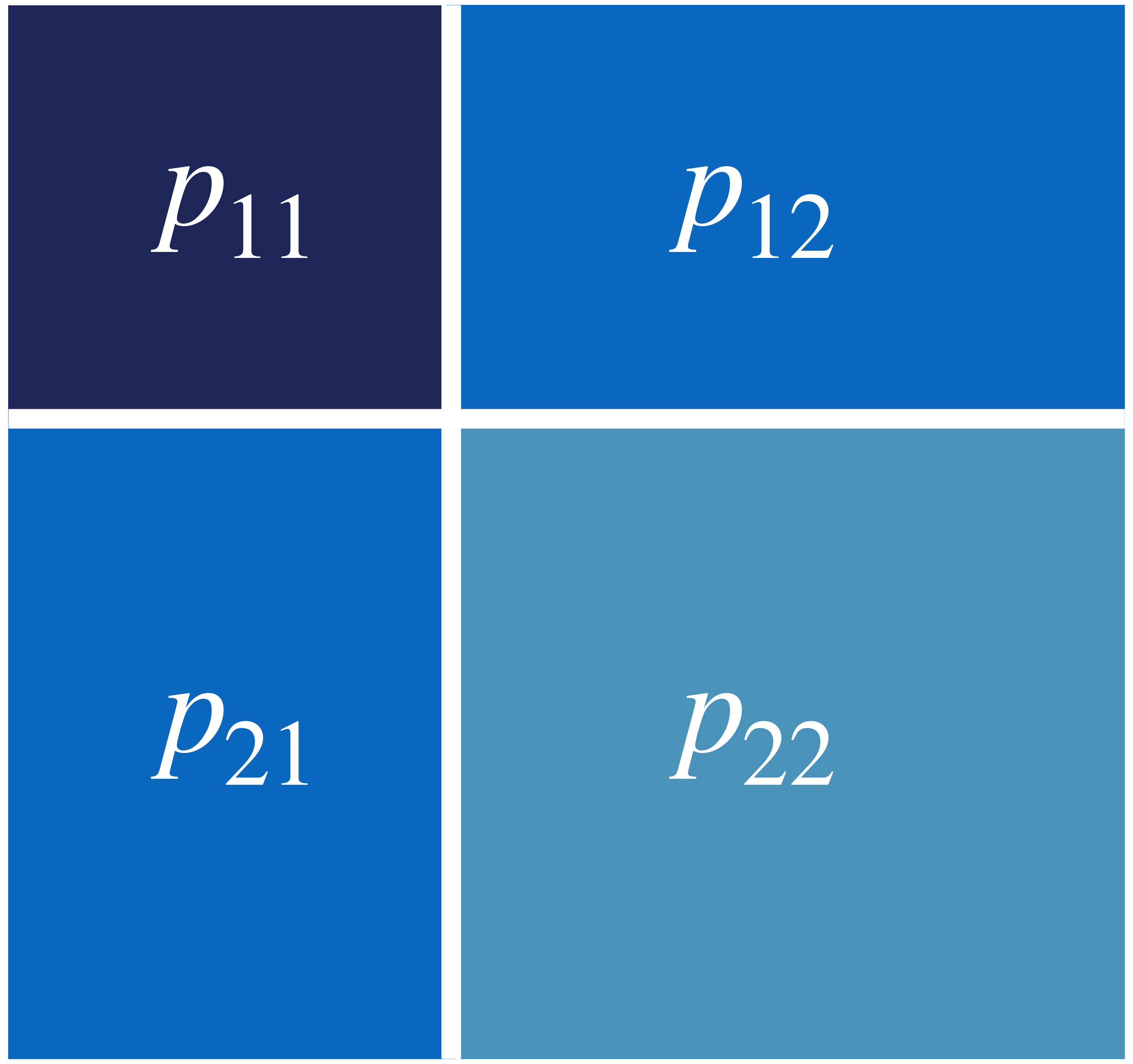

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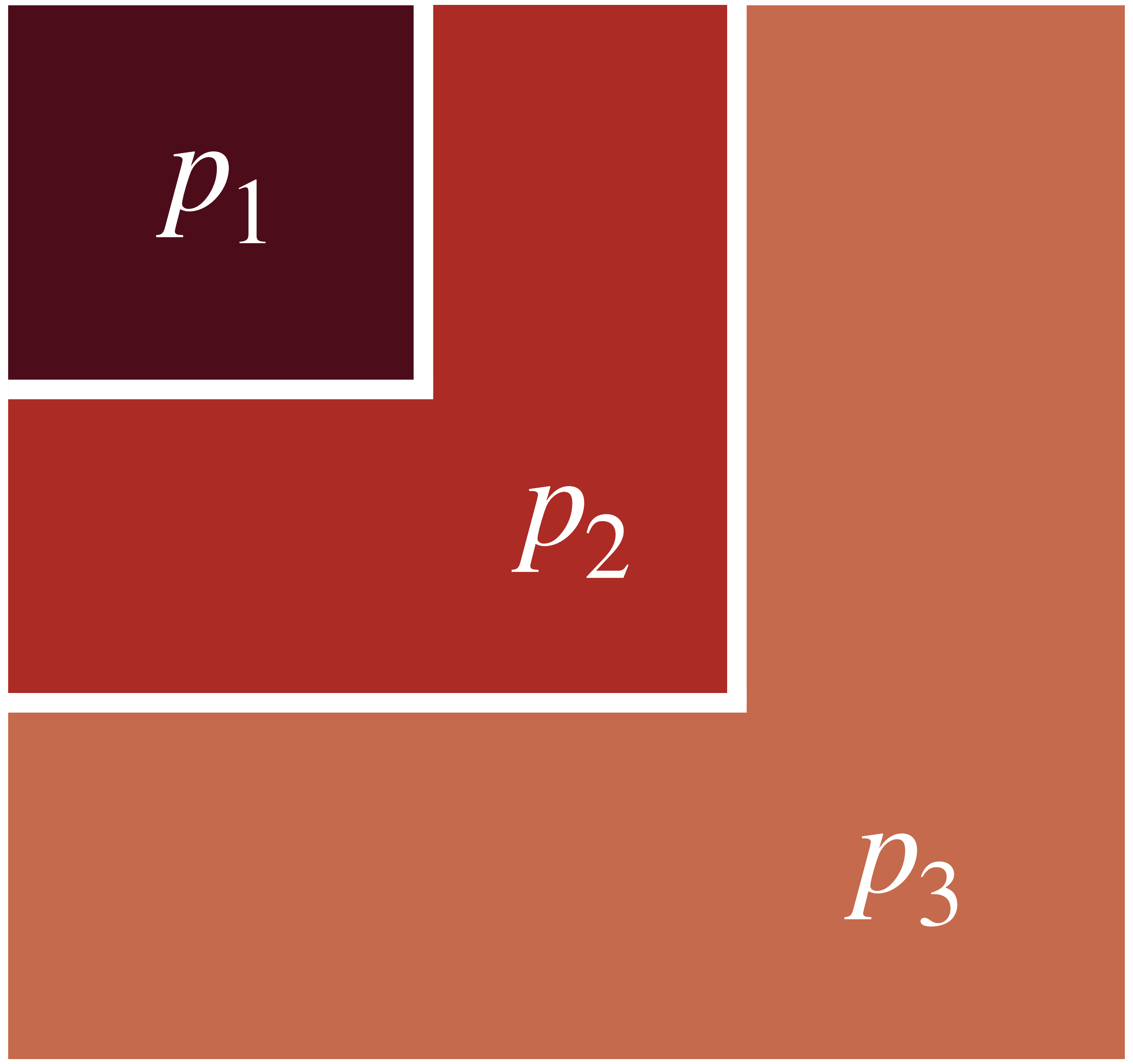

$$p_1 > p_2 > \ldots > p_\ell$$

Model Selection and Description Length

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Posterior odds ratio

$$\Lambda = \frac{P\left(\hat{\theta}_{\mathscr{H}}, \mathscr{H} \mid \mathbf{A}\right)}{P\left(\hat{\theta}_{\mathscr{L}}, \mathscr{L} \mid \mathbf{A}\right)} > 1$$

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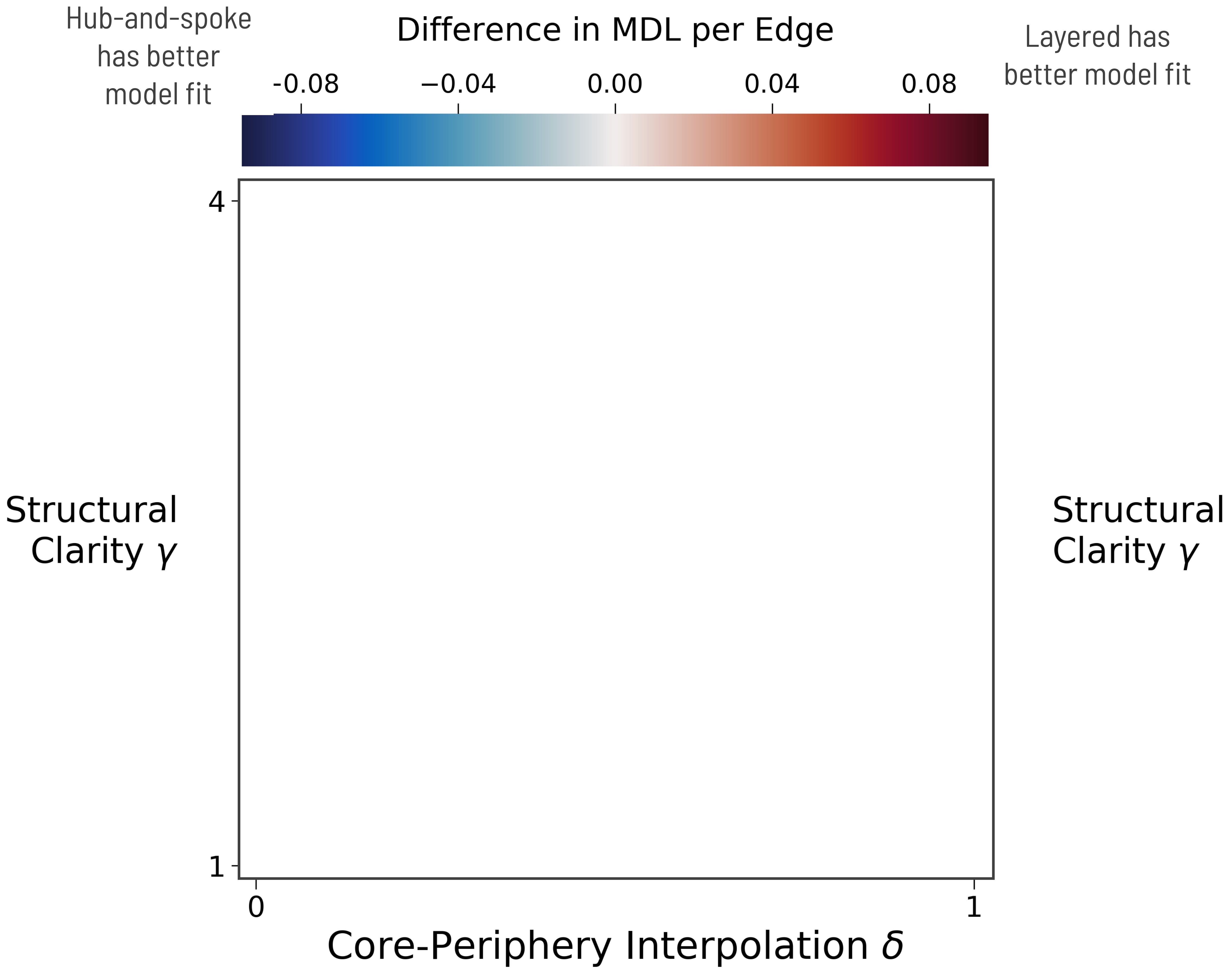
The smaller the description length, the better the model fit

Synthetic Validation: Discerning Models



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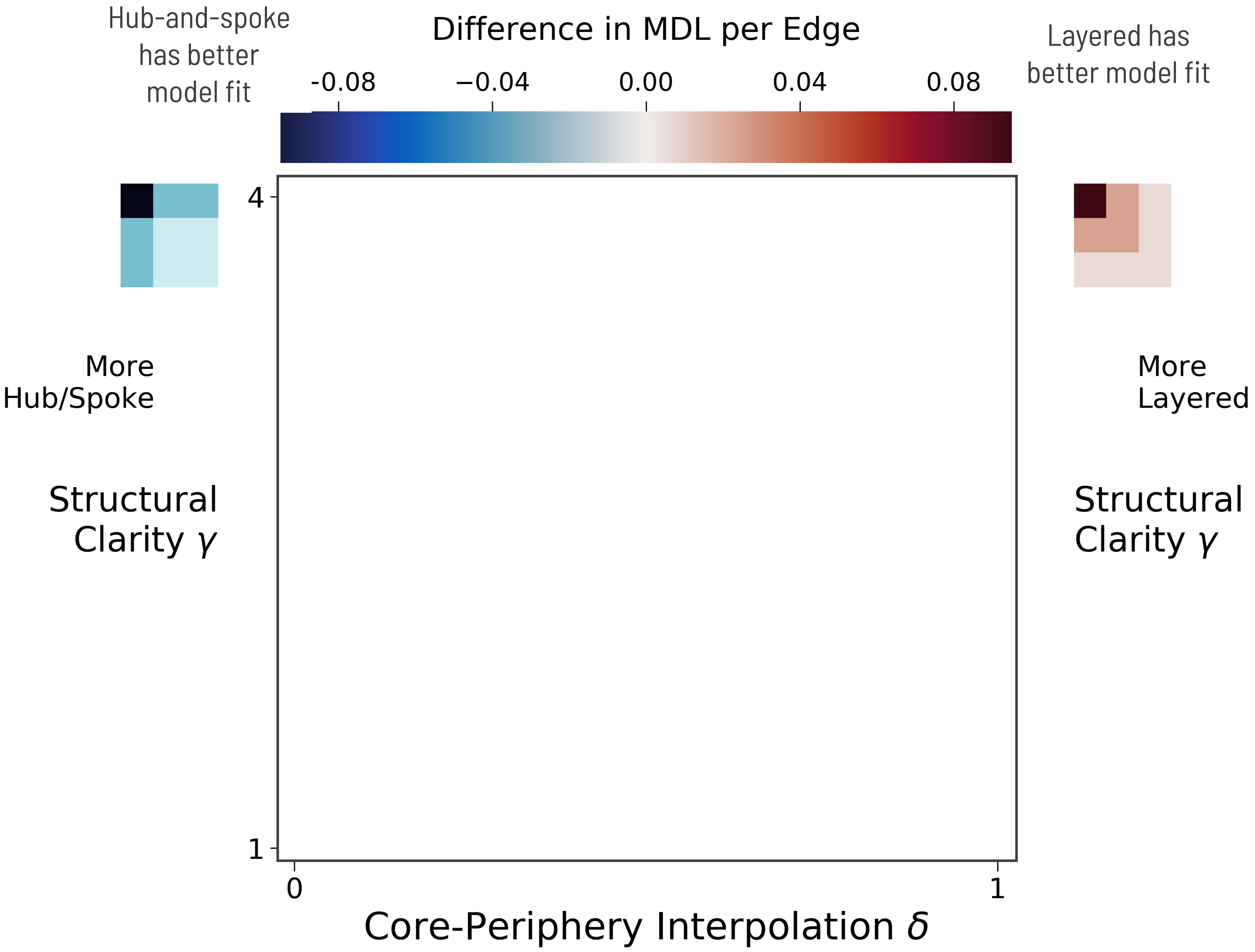

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- $\delta = 0$, hub-and-spoke structure
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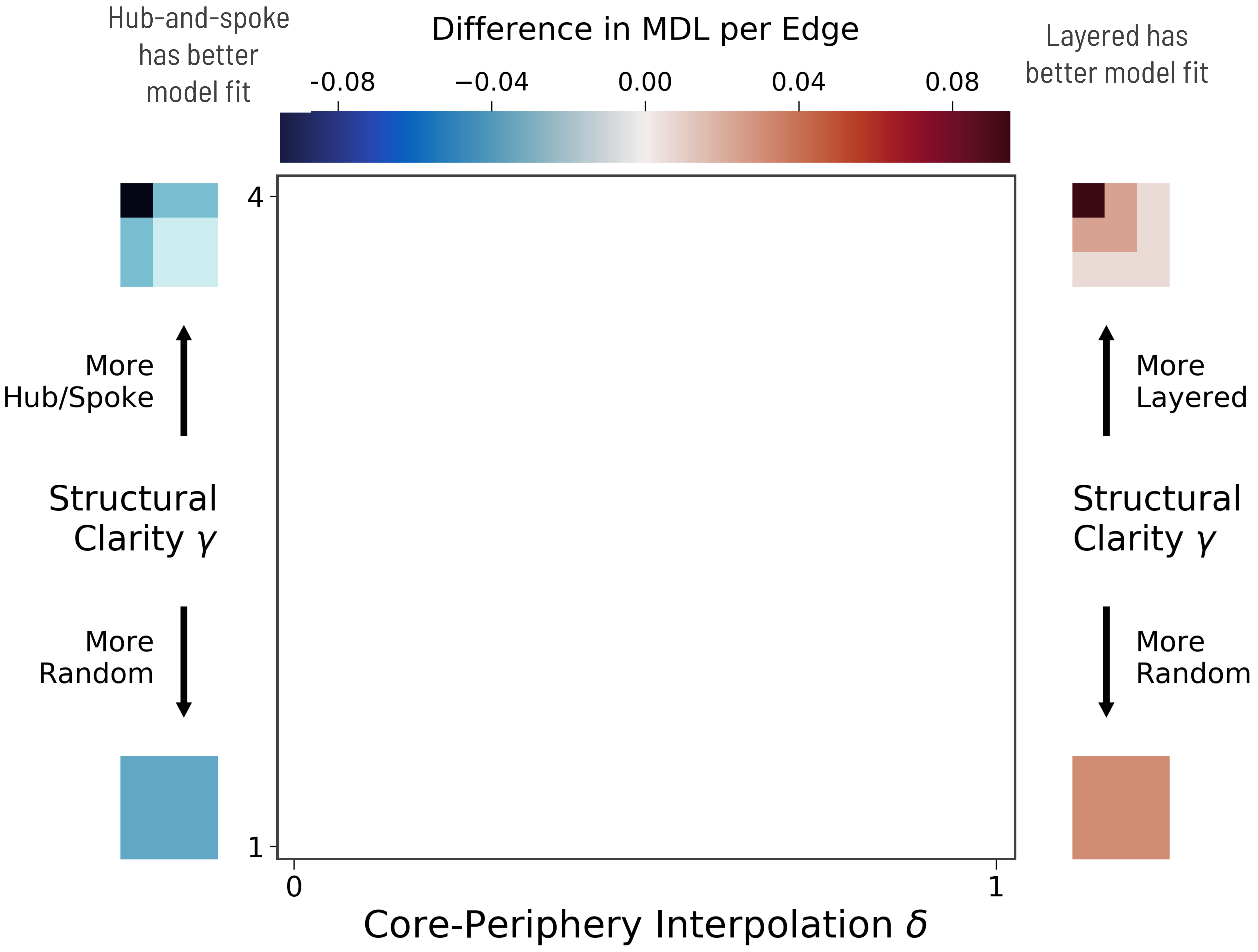

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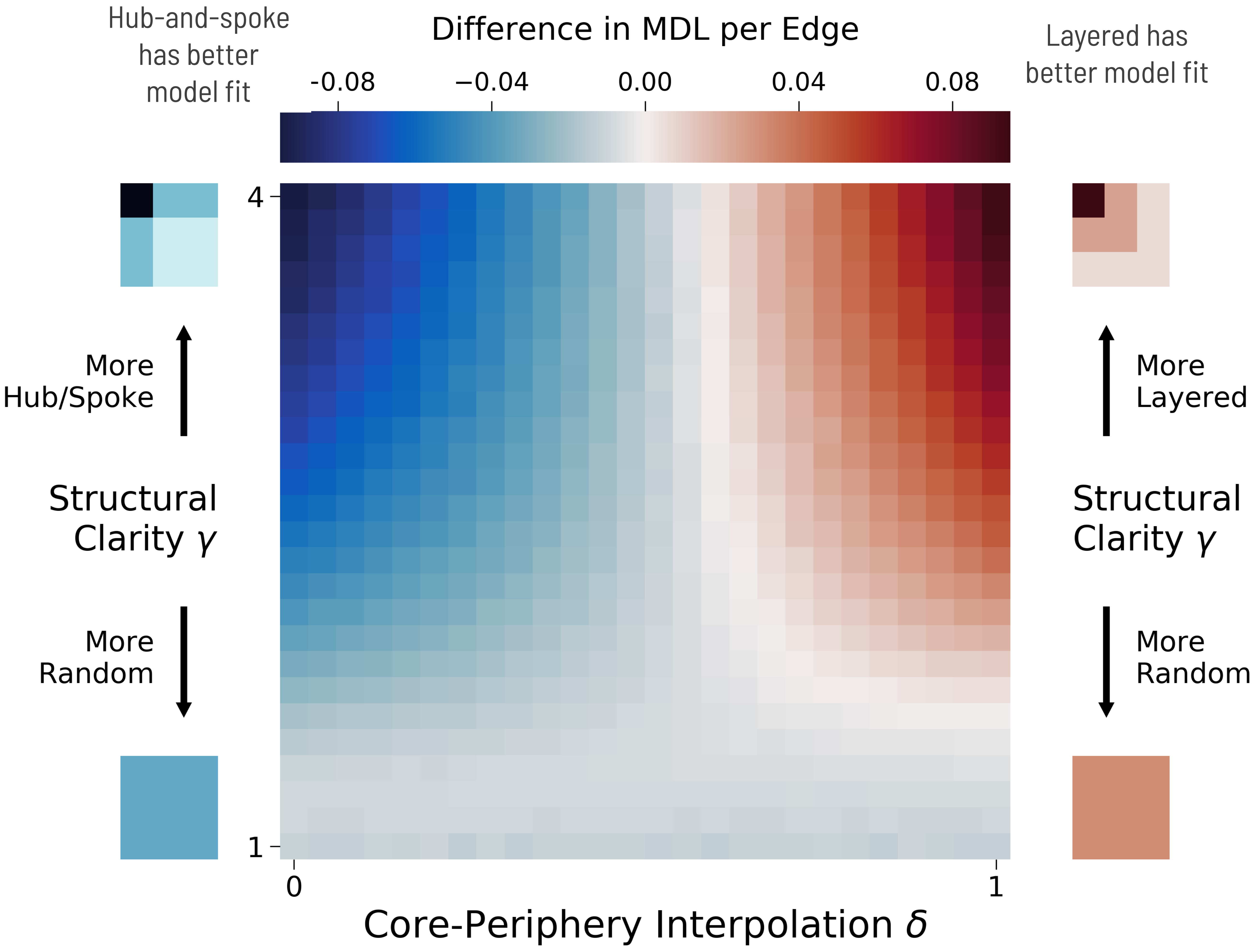

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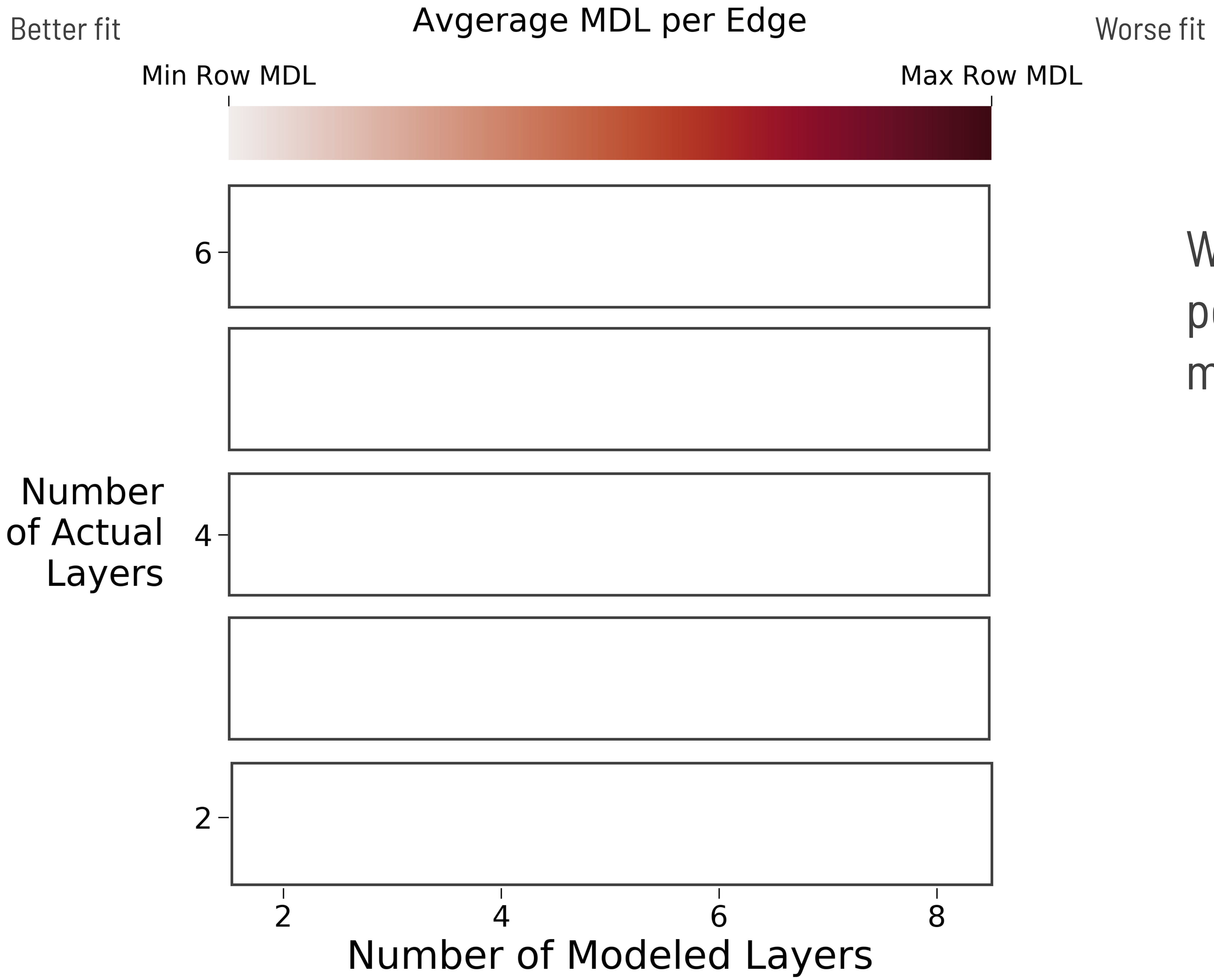

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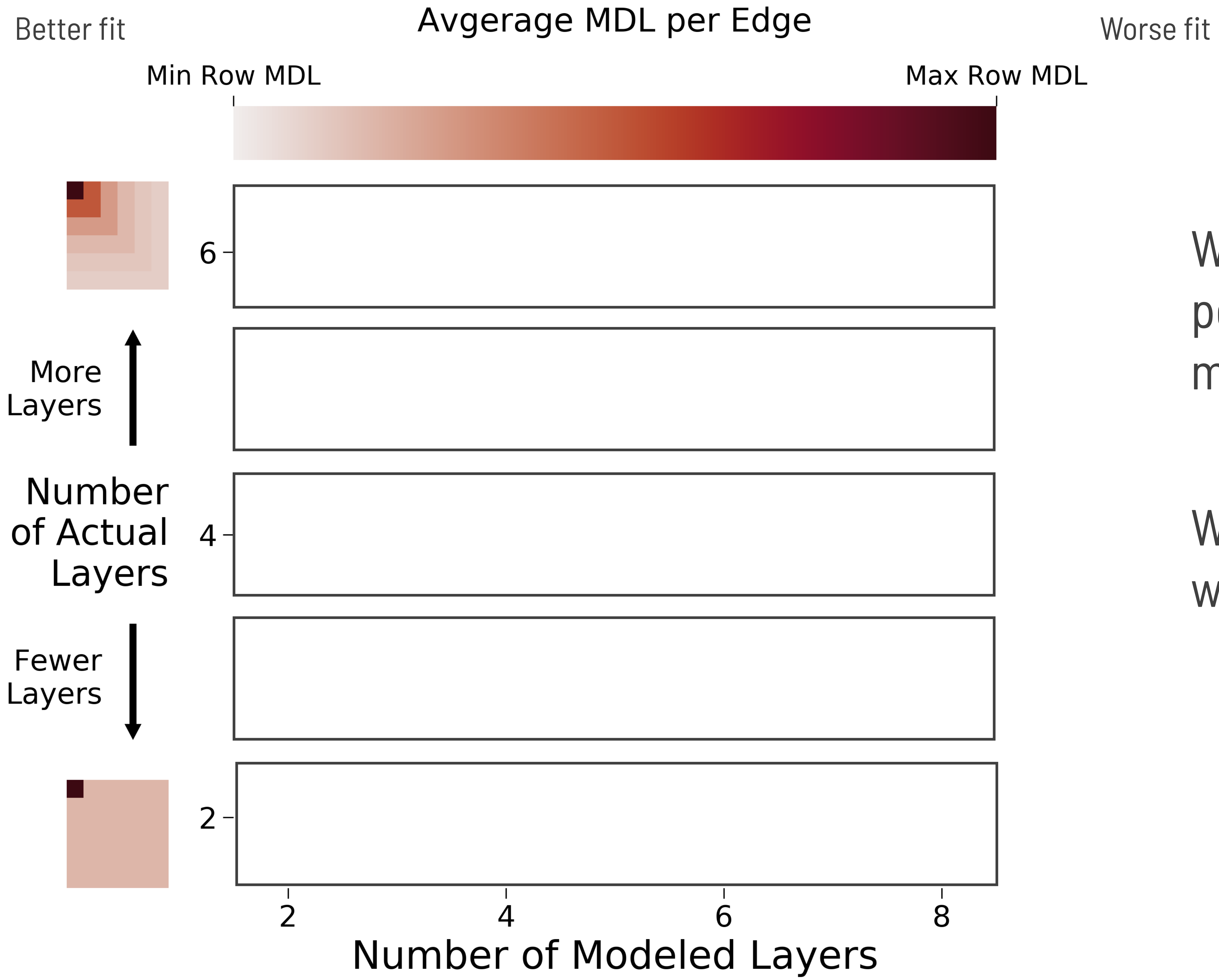

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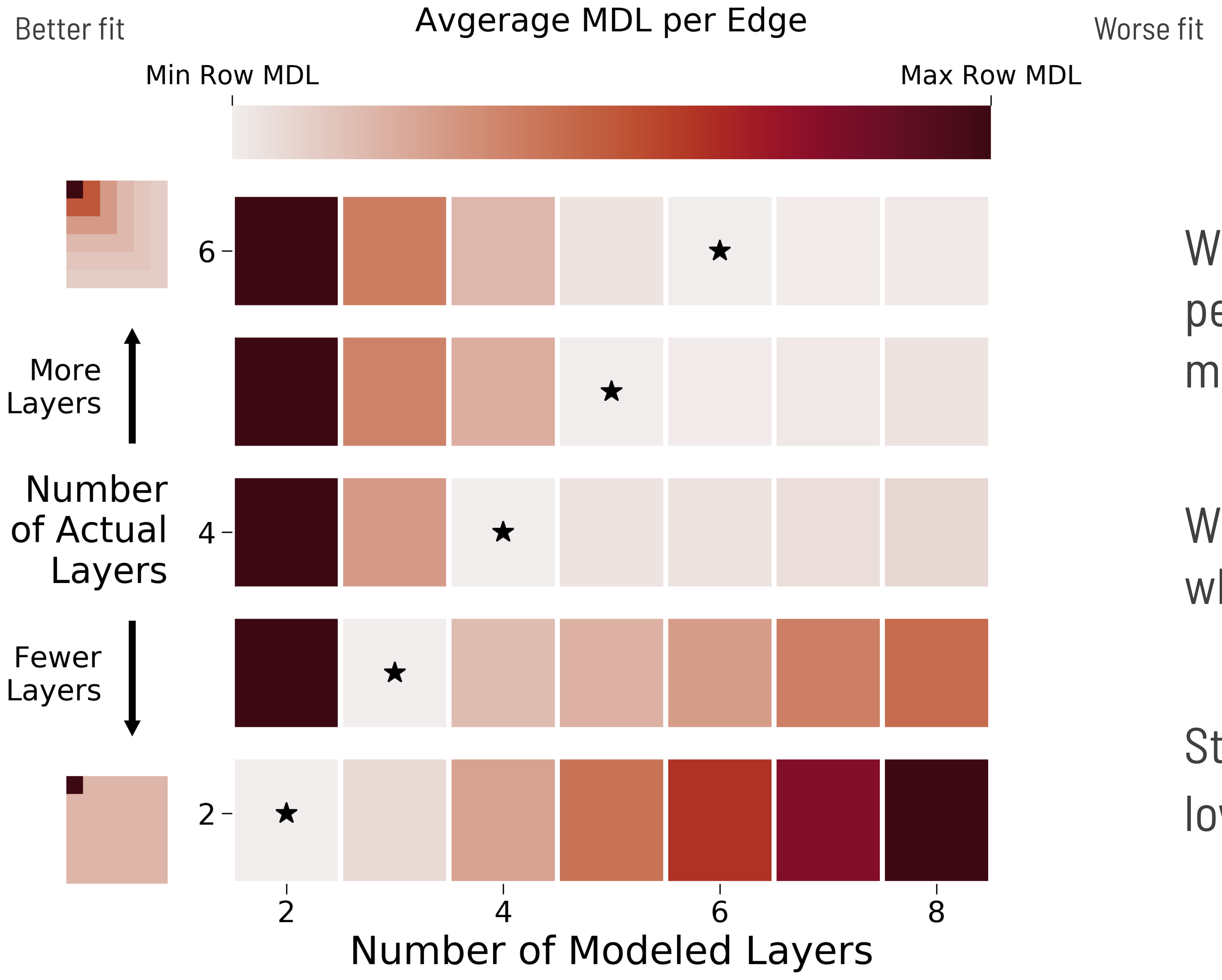

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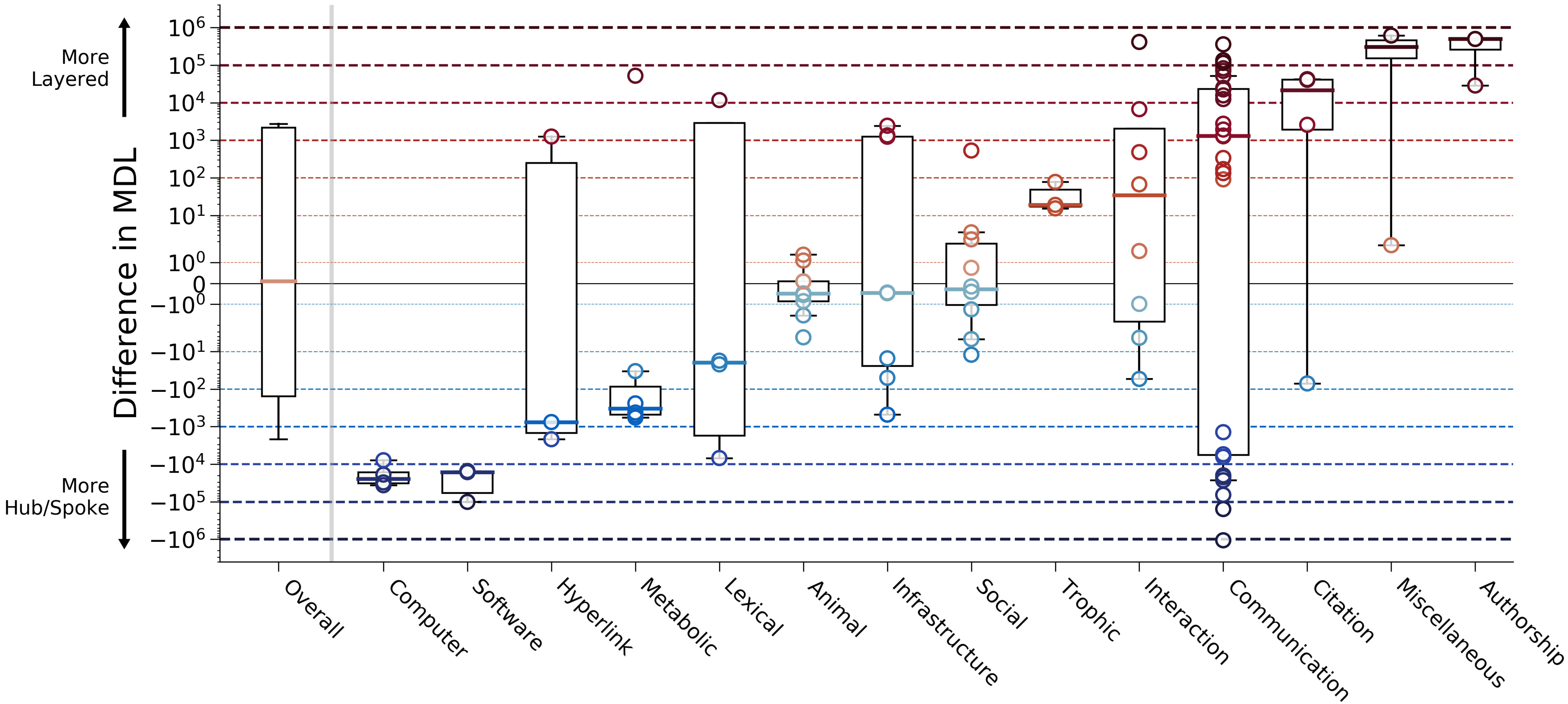
Stars (★) in each row indicate the model with the lowest description length on average

Diversity of Core-Periphery Structure



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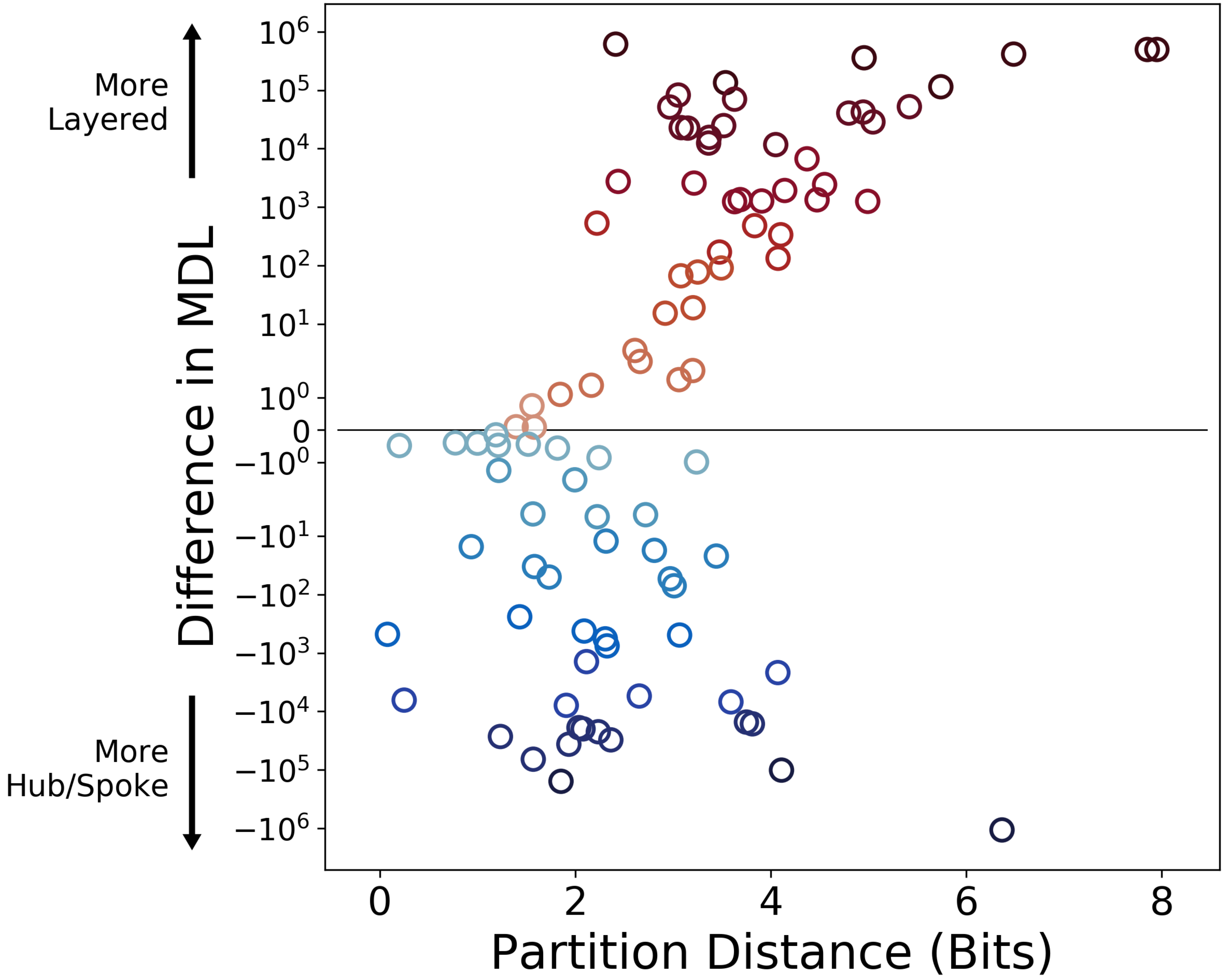
Partition Dissimilarity is Explained by the Core-Periphery Typology



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A Clarified Typology of Core-Periphery Structure

1. The two most popular core-periphery algorithms, the two-block model and the *k*-cores decomposition, give inconsistent descriptions of core-periphery structure

2. We have proposed a clarified typology of core-periphery structure:
   There are hub-and-spoke and layered core-periphery structures

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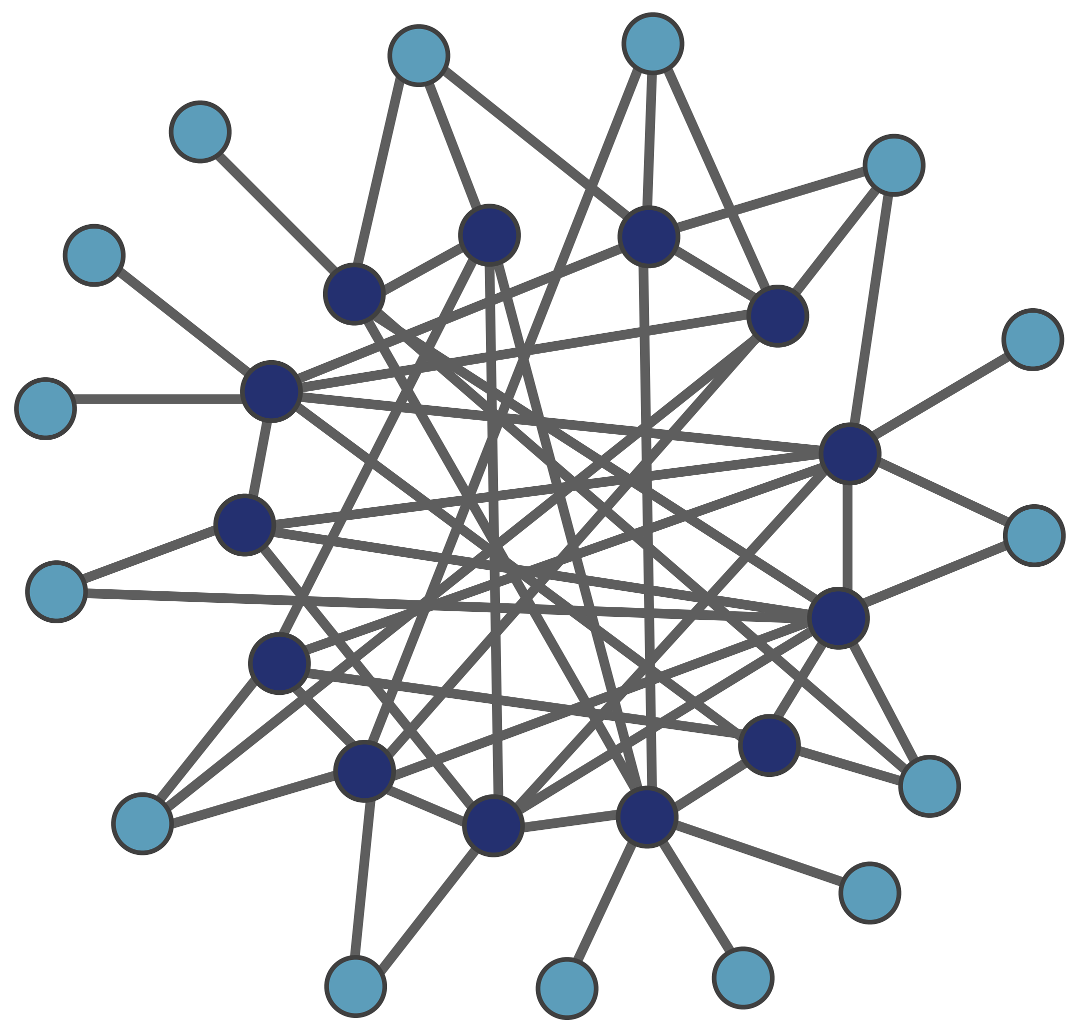

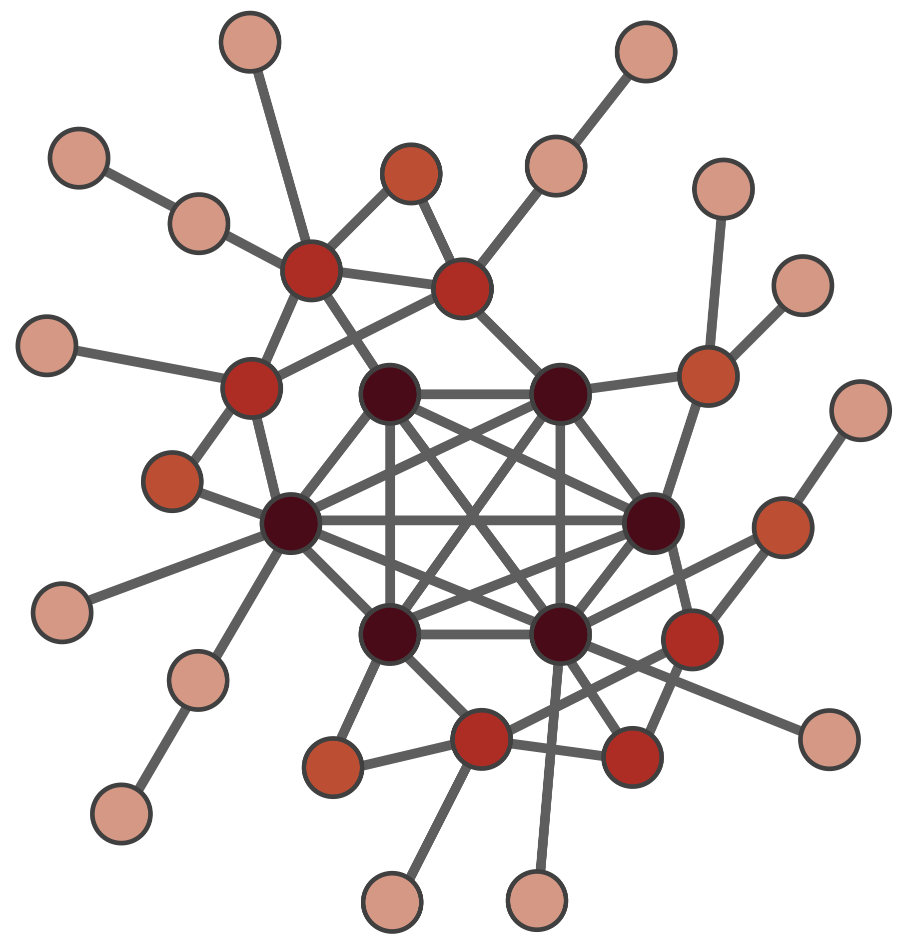

Collaborators

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Associate Professor

Communication Studies
Network Science Institute
Northeastern University

netsi | Northeastern University
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CoMM Lab

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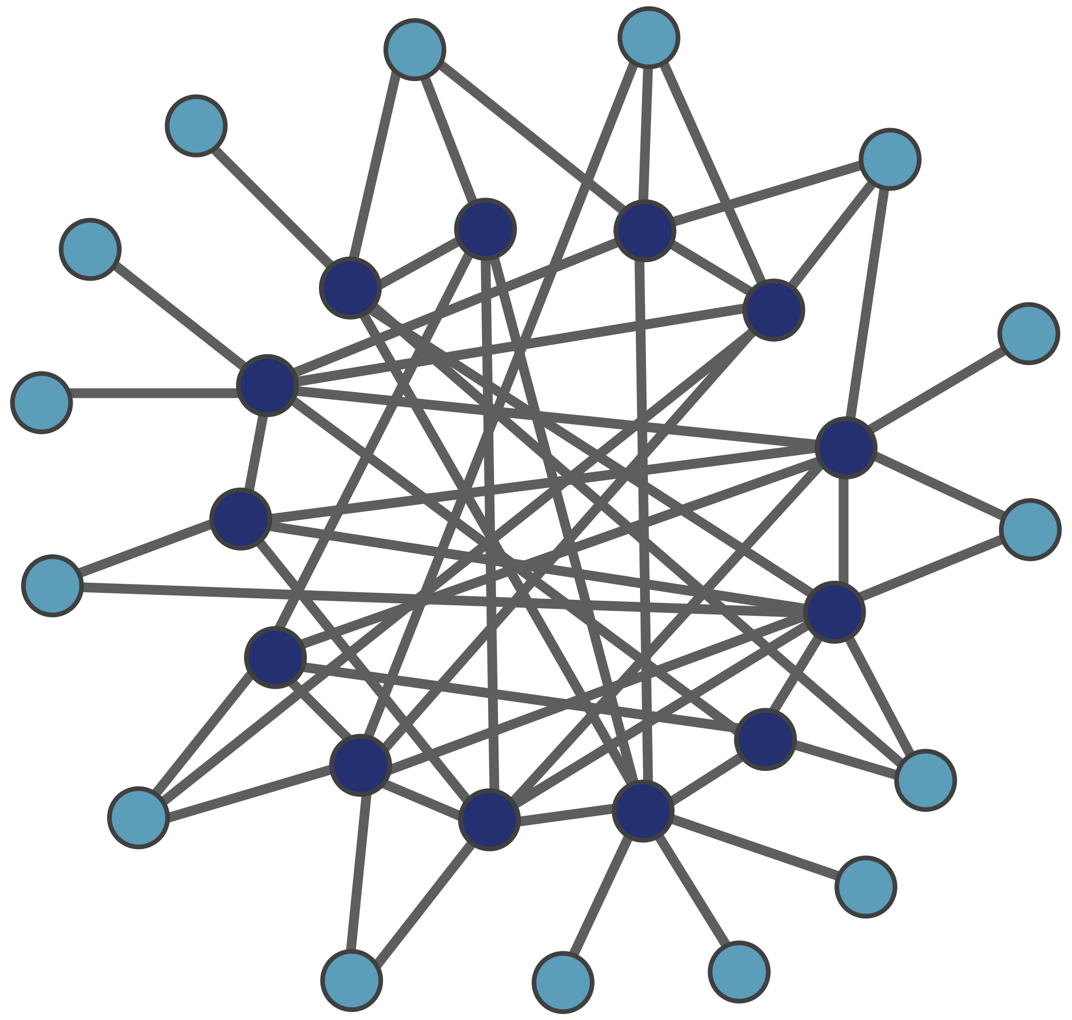

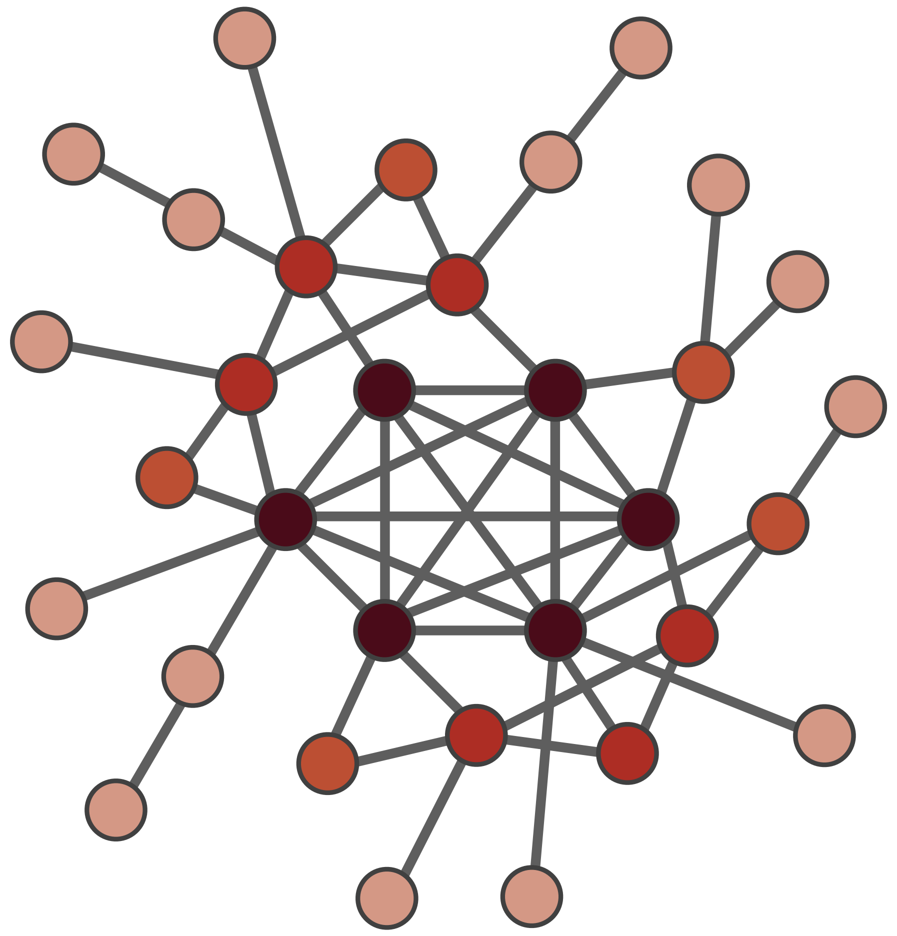

Thank you for your time!

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